Inversion of sampled-data system approximates the continuous-time counterpart in a noncausal framework ⋆

Takuya Sogo [a],

[a] *Department of Mechanical Engineering, Chubu University, Aichi 487-8501, Japan*

Abstract

Although correspondence between the poles of a continuous-time and sampled-data system with a piecewise constant input is simple and desirable from the stability viewpoint, the relationship between zeros is intricate. Inversion of a sampled-data system is mostly unstable irrespective of the stability of the continuous-time counterpart. This makes it difficult to apply inversion-based control techniques such as perfect tracking, transient response shaping or iterative learning control to sampled-data systems. Although recently developed noncausal inversion techniques help us to circumvent unboundedness of the inversion caused by unstable zeros, whether the inversion of sampled-data systems approximates the continuous-time counterpart or not as the sample period is shortened is still to be determined. This article gives a positive conclusion to this problem.

Key words: sampled-data systems; inverse system; transmission zeros; discrete-event dynamic system; feedforward control; nonminimum phase systems; zero-order hold; learning control

1 Introduction

In recent times, control systems have typically been installed in digital devices that include samplers and zero-order holds, which convert continuous-time signals into discrete-time signals and vice versa. A zero-order hold generates piecewise constant functions that can approximate any uniformly continuous function $u(t)$ defined on the infinite time horizon, i.e. $\|u(\lfloor t/\tau\rfloor\tau) - u(t)\|_\infty \to 0$ as the sample period $\tau \to 0$ where $\|x(t)\|_\infty = \sup\{|x(t)|; t \in (-\infty, +\infty)\}$ and $\lfloor t/\tau\rfloor$ denotes the maximum integer that does not exceed t/τ. This implies that $\left\| \int_{-\infty}^{+\infty} g(t-\sigma)u(\lfloor\sigma/\tau\rfloor\tau)d\sigma - \int_{-\infty}^{+\infty} g(t-\sigma)u(\sigma)d\sigma \right\|_\infty \to 0$ for any stable linear system g, i.e. the output of stable continuous-time systems with a piecewise constant input $u(\lfloor t/\tau\rfloor\tau)$ approximates the output of the same systems with a continuous input $u(t)$ as the sample period is shortened. This fact encourages us to replace analog controllers with digital controllers with a sufficiently small sample time. In contrast to the earlier mentioned convenient properties, it is recognized that there is no simple correspondence between inversion of the system with continuous input and piecewise constant input. This fact is highlighted by investigations from the viewpoint of transfer functions. Consider a linear causal system with an impulse response $g(t)$. Then, the transfer function is $G(s) = \mathcal{L}[\int_{-\infty}^{+\infty} g(t-\sigma)u(\sigma)d\sigma]/\mathcal{L}[u(t)]$ where $\mathcal{L}$ denotes the one-sided Laplace transform. Assume that the transfer function is expressed as

$$G(s) = \frac{K(s-\gamma_1)(s-\gamma_2)\cdots(s-\gamma_m)}{(s-p_1)(s-p_2)\cdots(s-p_n)} \tag{1}$$

or $G(s) = c(sI-A)^{-1}b$, where (A, b, c) is a state space representation. Then, the discrete-time transfer function of the system with piecewise constant inputs $u(\lfloor\sigma/\tau\rfloor\tau)$ on the sample time $t = k\tau$ $(k = 0, \pm1, \cdots)$ is $H_\tau(z) = \mathcal{Z}[\int_{-\infty}^{+\infty} g(k\tau-\sigma)u(\lfloor\sigma/\tau\rfloor\tau)d\sigma]/\mathcal{Z}[u(k\tau)]$; equivalently, $H_\tau(z) = \mathcal{Z}[\mathcal{L}^{-1}[G(s)\mathcal{L}[u(\lfloor t/\tau\rfloor\tau)]](k\tau)]/\mathcal{Z}[u(k\tau)]$, which is expressed as

$$H_\tau(z) = \frac{cb_\tau(z-q_1(\tau))\cdots(z-q_{n-1}(\tau))}{(z-\exp(p_1\tau))\cdots(z-\exp(p_n\tau))} \tag{2}$$

or $H_\tau(z) = c(zI-A_\tau)^{-1}b_\tau$, where $\mathcal{Z}$ is the one-sided z-transform, $A_\tau = \exp(A\tau)$ and $b_\tau = \int_0^\tau \exp(At)bdt$.

Although correspondence between the poles of $G(s)$ and $H_\tau(z)$ is simple and desirable from the stability viewpoint, the relationship between zeros is intricate. It is

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Corresponding author T. Sogo. Tel. +81-568519771; fax +81-568511194.

Email address: sogo@isc.chubu.ac.jp (Takuya Sogo).

Table 1
Zeros of Euler–Frobenius polynomial $B_{n-m}(z)$

$n-m$	zeros
2	-1
3	$-2-\sqrt{3}$, $1/(-2-\sqrt{3})$
4	$-5-2\sqrt{6}$, -1, $1/(-5-2\sqrt{6})$
5	λ_{51}, λ_{52}, $1/\lambda_{52}$, $1/\lambda_{51}$ ($\lambda_{51} \approx -23$, $\lambda_{52} \approx -2.3$)
6	λ_{61}, λ_{62}, -1, $1/\lambda_{62}$,$1/\lambda_{61}$ ($\lambda_{61} \approx -51$, $\lambda_{62} \approx -4.5$)
$\vdots$	$\vdots$
odd $i-1$	$\lambda_{i-11}, \cdots, \lambda_{i-1(i-2)/2}, 1/\lambda_{i-1(i-2)/2}, \cdots, 1/\lambda_{i-11}$ ($\lambda_{i-11} < \cdots < \lambda_{i-1(i-2)/2} < -1$)
even i	$\lambda_{i1}, \cdots, \lambda_{i(i-2)/2}, -1, 1/\lambda_{i(i-2)/2}, \cdots, 1/\lambda_{i1}$ ($\lambda_{i1} < \cdots < \lambda_{i(i-2)/2} < -1$)

known that the zeros of $H_\tau(z)$ have the following asymptotic properties in terms of the sample period τ [1,7]: $q_i(\tau) = 1 + \gamma_i \tau + O(\tau^2)(i = 1, \cdots, m)$ and $q_i(\tau) \to$ zeros of $B_{n-m}(z)$ $(i = m+1, \cdots, n-1)$ as $\tau \to 0$, where $B_{n-m}(z)$ is the Euler–Frobenius polynomial, the zeros of which are located on the negative real axis symmetrically with respect to -1 (Table 1)[4,19]. This implies that inversion of the discrete-time system $1/H_\tau(z)$ with a small sample period is mostly unstable even if the continuous-time counterpart $1/G(s)$ is stable.

On the other hand, system inversion plays crucial roles in many control applications such as perfect tracking, transient response shaping, disturbance attenuation, and noise cancellation. The aforementioned fact makes it difficult to apply inversion-based control techniques developed for continuous-time systems to sampled-data systems with piecewise constant inputs. For example, consider a shaping transient response of $G(s)$. Then, as long as the zeros of $G(s)$ are stable, one can employ $M(s)/G(s)$ as a prefilter of $G(s)$, where $M(s)$ is a model that has a desired response. However, it is not necessarily possible to apply a corresponding approach directly to the case of piecewise constant inputs because the discrete-time prefilter $N_\tau(z)/H_\tau(z)$ is mostly unstable; here, $N_\tau(z)$ is the discrete-time counterpart of $M(s)$. Nonetheless, one can avoid unboundedness of the prefilter by introducing a discrete-time version of the so-called stable inversion technique, which is a method to apply anticausal convolution to antistable parts of the inverse system and generate bounded outputs[3,9]. Still, even though one circumvents the unboundedness due to unstable zeros of $H_\tau(z)$, whether the discrete-time prefilter $N_\tau(z)/H_\tau(z)$ can be substituted for the continuous-time prefilter $M(s)/G(s)$ is still a question. Recall that $H_\tau(z)$ approximates $G(s)$ for uniformly continuous functions $u(t)$. In such a case, whether $1/H_\tau(z)$ approximates $1/G(s)$ or not must be determined. In this article, the author presents an affirmative conclusion to this problem.

This article is organized as follows: Section 2 defines noncausal stable inversion with the two-sided Laplace transform and z-transform and formulates the main problem with illustrative numerical examples; Section 3 demonstrates the main results on an approximation in the noncausal framework; and Section 4 concludes the work.

2 Noncausal inversion and formulation of the approximation problem

Since feedback control is essentially causal, the one-sided Laplace transforms and the one-sided z-transforms have been widely used as mathematical tools to analyze and design linear feedback controllers. In this framework, transfer functions with unstable poles that are located in the right half plane for continuous-time systems or outside the unit circle for discrete-time systems correspond to diverging signals. This implies that the inverse of systems with unstable zeros is of no practical use. However, feedforward control is not necessarily causal in applications such as perfect tracking, transient response shaping or iterative learning control. Noncausal feedforward control has been proposed to achieve better tracking than that given by causal controllers [8–11]. It is known that noncausality enlarges the application scope of iterative learning control[12,17,18]. In this work, the author introduces the two-sided Laplace transform and the z-transform as mathematical tools to analyze noncausal inversion.

The two-sided Laplace transform of a function $f(t)$ where $t \in (-\infty, +\infty)$ is defined as $\mathcal{L}[f(t)](s) = F(s) = \int_{-\infty}^{+\infty} e^{-st} f(t)dt$, which is an analytic function of $s \in \mathbb{C}$ in the vertical strip area $\gamma_1 < \mathrm{Re}(s) < \gamma_2$[16]. Let α be a real number satisfying $\gamma_1 < \alpha < \gamma_2$. The inverse Laplace transform is then expressed by

$$\mathcal{L}^{-1}[F(s)](t) = f(t) = \frac{1}{2\pi j}\int_{\alpha - j\infty}^{\alpha + j\infty} e^{st}F(s)ds$$
$$= \begin{cases} \sum_{\mathrm{Re}(p_n)<\alpha} \mathrm{Res}(e^{st}F(s), p_n) & t \geq 0 \\ \sum_{\mathrm{Re}(p_m)>\alpha} \mathrm{Res}(-e^{st}F(s), p_m) & t < 0 \end{cases} \quad (3)$$

where $\{p_n\}$ and $\{p_m\}$ are the sets of poles of $F(s)$. For example, consider a bounded function $f(t)$ defied as $f(t) = e^{-t}$ for $t \geq 0$ and $f(t) = e^{2t}$ for $t < 0$. Then, we have $F(s) = 1/(s+1) + 1/(2-s)$, which is analytic on $\{s; -1 < \mathrm{Re}(s) < 2\}$.

The two-sided z-transform of a discrete-time function $h(k)$ where $k \in \mathbb{Z}$ is defined as $\mathcal{Z}[h(k)](z) = H(z) = \sum_{k=-\infty}^{+\infty} h(k)z^{-k}$, which is an analytic function of $z \in \mathbb{C}$ in the annular domain $r_0 < |z| < R_0$. Let α be a positive real number satisfying $r_0 < \alpha < R_0$. Then, the inverse

z-transform is expressed by

$$\mathcal{Z}^{-1}[H(z)](k) = h(k) = \frac{1}{2\pi j}\oint_{|z|=\alpha} z^k H(z)\frac{dz}{z}$$
$$= \begin{cases} \sum_{|p_n|<\alpha} \mathrm{Res}(z^{k-1}H(z), p_n) & k \le 0 \\ \sum_{|p_m|>\alpha} \mathrm{Res}(-z^{k-1}H(z), p_m) & k < 0 \end{cases} \quad (4)$$

For example, consider a series $\{h(k); k \in \mathbb{Z}\}$ defined as $h(k) = 1/2^k$ for $k \ge 0$ and $h(k) = 3^k$ for $k < 0$. Then, $H(z) = 2z/(2z-1) + z/(3-z)$ which is analytic $1/2 < |z| < 3$.

Consider again the linear causal systems $G(s)$ and $H_\tau(z)$, which are assumed to be stable. It should be noted that these transfer functions are unchanged although they are defined by the two-sided transforms. Consider an input function $u(t)$ that achieves perfect tracking of a desired trajectory $y_d(t)$, i.e. $\mathcal{L}[y_d(t)] = G(s)\mathcal{L}[u(t)]$. Then, $u(t)$ is expressed as

$$u(t) = \mathcal{L}^{-1}[1/G(s)\mathcal{L}[y_d(t)]](t) \quad (5)$$

If $1/G(s)$ has poles in the right half plane where it is assumed that $\alpha = 0$ in (3), the function (5) varies depending on whether the transform is one-sided or two-sided; for the latter case, the convolution of (5) is noncausal with respect to $y_d(t)$; the mapping from y_d to u is equivalent to the stable inversion that has been proposed in terms of the state space representation [9]. Next, consider an input sequence $\{\bar{u}(k); k \in \mathbb{Z}\}$ that generates a piecewise constant input $\bar{u}(\lfloor t/\tau \rfloor)$ that makes the output equal to the desired trajectory at the sample time $t = k\tau$; in other words, $y_d(k\tau) = \mathcal{L}^{-1}[G(s)\mathcal{L}[\bar{u}(\lfloor t/\tau \rfloor)]](k\tau)$, and equivalently, $\mathcal{Z}[y_d(k\tau)] = H_\tau(z)\mathcal{Z}[\bar{u}(k)]$. Then, $\bar{u}(k)$ is expressed as

$$\bar{u}(k) = \mathcal{Z}^{-1}[1/H_\tau(z)\mathcal{Z}[y_d(k\tau)]](k) \quad (6)$$

This is noncausal convolution if the transform is two-sided and $1/H_\tau(z)$ has poles outside the unit circle where it is assumed that $\alpha = 1$ in (4).

Example 1 *Consider* $G(s) = \frac{100(s+5)}{(s+1)(s+2)(s+3)(s+4)}$. *Then,*

$$H_\tau = \frac{1.1151(z+1.979)(z+0.1415)(z-0.08225)}{(z-0.6065)(z-0.3679)(z-0.2231)(z-0.1353)} \quad (7)$$

$$H_\tau = \frac{0.19036(z+2.723)(z+0.1961)(z-0.2865)}{(z-0.7788)(z-0.6065)(z-0.4724)(z-0.3679)} \quad (8)$$

for $\tau = 0.5$ *or* 0.25, *respectively. As a desired trajectory, we consider a function defined as* $y_d(t) = f((t-5)^2)$ *for* $t \in (4, 6)$ *and* $y_d(t) = 0$ *for* $t \in [0, 4]$ *or* $[6, 10]$, *where* $f(x) = -x^5 + 5x^4 - 10x^3 + 10x^2 - 5x + 1$. *The inversions (6) for causal and noncausal cases are compared.Since transfer functions (7) and (8) have a zero outside the unit circle, the convolution defined by (6) is causal or noncausal depending on whether* $1/H_\tau(z)$ *is considered as a function defined with the one-sided or two-sided z-transform, respectively.*

Figure 1 shows the results in the case of a causal inversion or one-sided z-transform with the sample period $\tau = 0.5$*; the lower plot presents piecewise constant input* $u(t) = \bar{u}(\lfloor t/\tau \rfloor)$ *obtained by (6) as the solid line; the upper plot presents response of* $G(s)$ *for this piecewise constant input, i.e.* $y(t) = \mathcal{L}^{-1}[G(s)\mathcal{L}[\bar{u}(\lfloor t/\tau \rfloor)]](t)$ *as the solid line and the desired trajectory* $y_d(t)$ *as the dashed line. Obviously, the inversion* $1/H_\tau(z)$ *as a discrete-time causal prefilter is unacceptable for practical applications because of its unboundedness on the infinite time horizon. However, this example shows that the inversion* $1/H_\tau(z)$ *is still unacceptable when it is used on the short time interval. It should be noted that although the desired trajectory* $y_d(t)$ *is recovered at the sample time* $t = k\tau$, *i.e.* $y(k\tau) = y_d(k\tau)$, *the output* $y(t)$ *on the inter-sample time is oscillating and far from the desired trajectory* $y_d(t)$. *This phenomenon is derived from the input* $u(t)$ *that includes diverging oscillation caused by the unstable zero of (7). This result conclusively shows that the inversion* $1/H_\tau(z)$ *as a discrete-time causal prefilter achieves perfect tracking only at the sample time but never at the inter-sample time.*

Figure 2 shows the results in the case of a noncausal inversion or two-sided z-transform with $\tau = 0.5$ *or* 0.25*; the lower and upper plots present the results similarly except that* $\mathcal{Z}$ *is the two-sided z-transform. The plots show that the desired trajectory* $y_d(t)$ *is recovered not only at the sample time but also approximately at the inter-sample time.*

Remark 2 *In the case of the two-sided z-transform, the calculation of the convolution (6) requires a summation from* $-\infty$ *to* $+\infty$. *However, since* $\mathcal{Z}^{-1}[1/H_\tau(z)](k) \to 0$ *and* $y_d(k\tau) \to 0$ *as* $k \to \pm\infty$, *the infinite convolution can be approximated with the finite one for a sufficiently long time interval; this was* $[0, 10]$ *in Example 1.*

In Fig. 2, it is observed that the output $y(t)$ appears to approach the desired trajectory $y_d(t)$ uniformly as the sample period τ tends to 0. Note that $y_d(t) = \mathcal{L}^{-1}[G(s)/G(s)\mathcal{L}[y_d(t)]](t)$. Then, one conjectures that the piecewise constant function $u(t) = \bar{u}(\lfloor t/\tau \rfloor)$ defined by the two-sided z-transform and (6) approaches the input function (5) or, in other words, $1/H_\tau(z)$ approximates $1/G(s)$ in the noncausal framework. Since $1/H_\tau(z)$ and $1/G(s)$ are not proper, it is conjectured that the approximation holds true if there exists derivatives of the desired trajectory y_d until an order that depends on the the zeros of the Euler–Frobenius polynomials (Table 1) and the relative degree of $G(s)$. Moreover, $H_\tau(z)$ has a zero approaching -1 when the relative degree is even. From the definition

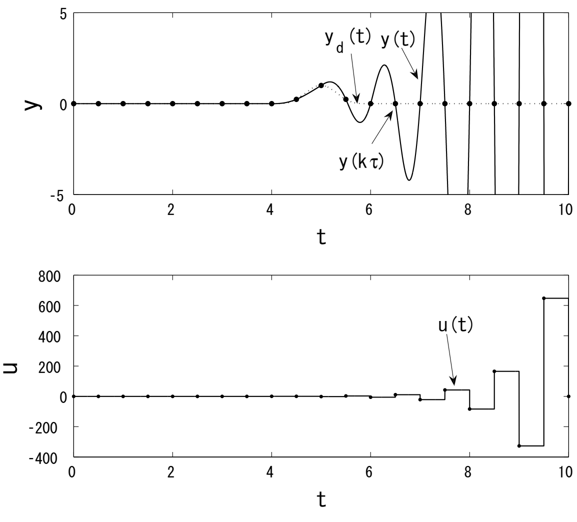

Fig. 1. Plots for the causal inversion of Example 1

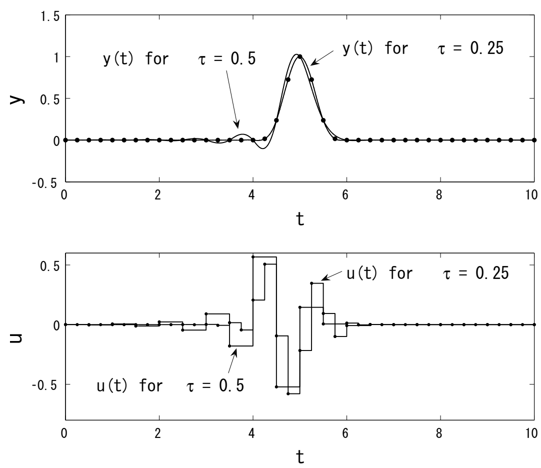

Fig. 2. Plots for the noncausal inversion of Example 1

(4), in such a case, the impulse response of $1/H_\tau(z)$ contains a component of $\{q(\tau)^k; k \geq 0\}$ for $|q(\tau)| \uparrow 1$ (as $\tau \to 0$) or $\{q(\tau)^k; k < 0\}$ for $|q(\tau)| \downarrow 1$ (as $\tau \to 0$), the convergence of which becomes slower with respect to the discrete-time indices. This makes it nontrivial to prove the conjecture.

Remark 3 *From the viewpoint of numerical analysis, the posed approximation problem is classified under discretization for integral equations of the first kind [5], which corresponds to the equation $y_d(t) = \mathcal{L}^{-1}[G(s)\mathcal{L}[u(t)]]$ in the settings. It is recognized that careful treatment is required to discretize this kind of integral equation [5,13]. Nonetheless, as far as the author knows, discretization based on the piecewise constant function, which is uncommon for numerical calculation but practical for real-time control applications, has not been researched thus far. This background encourages us to tackle the problem.*

3 The main result on approximation

In this section, the next theorem is proved.

Theorem 4 *Consider $u_d(t) = \mathcal{L}^{-1}[1/G(s)\mathcal{L}[y_d(t)]](t)$, which or the derivative of which is uniformly continuous on $(-\infty, +\infty)$ for $G(s)$ with an odd or even relative degree, respectively. Then,*

$$\left\|\mathcal{Z}^{-1}[1/H_\tau(z)\mathcal{Z}[y_d(k\tau)]](\lfloor t/\tau \rfloor) - u_d(t)\right\|_\infty \to 0 \quad (9)$$

as $\tau \to 0$ where the transforms are two-sided.

Remark 5 *For the continuous-time inversion $1/G(s)$, $u_d(t)$ or $du_d(t)/dt$ is continuous if and only if $y_d(t)$ is continuous with its derivative until the $(n-m)$th or $(n-m+1)$th order, respectively[14,15]. This fact is useful for choosing the desired trajectory $y_d(t)$ satisfying the condition of Theorem 4 for applications of the discrete-time inversion $1/H_\tau(z)$.*

Since we have $\bar{y}(t) = \mathcal{L}^{-1}[G(s)\mathcal{L}[u_d(\lfloor t/\tau \rfloor \tau)]](t)$ and $\mathcal{Z}^{-1}[1/H_\tau(z)\mathcal{Z}[\bar{y}(k\tau)]](k) = u_d(k\tau)$ from the definition of $H_\tau(z)$, we obtain

$$\begin{aligned}
&\left\|\mathcal{Z}^{-1}[1/H_\tau(z)\mathcal{Z}[y_d(k\tau)]](\lfloor t/\tau \rfloor) - u_d(t)\right\|_\infty \\
\leq &\left\|\mathcal{Z}^{-1}[1/H_\tau(z)\mathcal{Z}[y_d(k\tau)]](\lfloor t/\tau \rfloor) - u_d(\lfloor t/\tau \rfloor \tau)\right\|_\infty \\
&+ \|u_d(\lfloor t/\tau \rfloor \tau) - u_d(t)\|_\infty \\
\leq &\left\|\mathcal{Z}^{-1}[1/H_\tau(z)\mathcal{Z}[v_\tau(k\tau)]](\lfloor t/\tau \rfloor)\right\|_\infty \\
&+ \|u_d(\lfloor t/\tau \rfloor \tau) - u_d(t)\|_\infty \qquad (10)
\end{aligned}$$

where $v_\tau(t) = \mathcal{L}^{-1}[G(s)\mathcal{L}[u_d(t) - u_d(\lfloor t/\tau \rfloor \tau)]](t)$. Since $\|u_d(\lfloor t/\tau \rfloor \tau) - u_d(t)\|_\infty \to 0$ for uniformly continuous u_d, the approximation (9) is implied by

$$\left\|\mathcal{Z}^{-1}[1/H_\tau(z)\mathcal{Z}[v_\tau(k\tau)]](k)\right\|_\infty \to 0 \qquad (11)$$

This will be shown herein. To this end, we decompose $1/H_\tau(z)$ as $1/H_\tau(z) = J_\tau(z)K_\tau(z)$ where

$$\begin{aligned}
J_\tau(z) =& \frac{\tau^{n-m}}{cb_\tau}\frac{(z-\exp(p_1\tau))\cdots(z-\exp(p_m\tau))}{(z-q_1(\tau))\cdots(z-q_m(\tau))} \\
&\times \frac{1}{(z-q_{m+1}(\tau))\cdots(z-q_{n-1}(\tau))} \qquad (12)\\
K_\tau(z) =& \frac{(z-\exp(p_{m+1}\tau))\cdots(z-\exp(p_n\tau))}{\tau^{n-m}} \qquad (13)
\end{aligned}$$

for $G(s)$ with an odd relative degree and

$$J_\tau(z) = \frac{\tau^{n-m}}{cb_\tau}\frac{(z-\exp(p_1\tau))\cdots(z-\exp(p_m\tau))}{(z-q_1(\tau))\cdots(z-q_m(\tau))} \times \frac{\tau}{(z^2-q_{m+1}(\tau)^2)(z-q_{m+2}(\tau))\cdots(z-q_{n-1}(\tau))} \tag{14}$$

$$K_\tau(z) = \frac{(z-\exp(p_{m+1}\tau))\cdots(z-\exp(p_n\tau))(z+q_{m+1}(\tau))}{\tau^{n-m+1}} \tag{15}$$

for $G(s)$ with an even relative degree. Without loss of generality, it is assumed $q_{m+1}(\tau) \to -1$ (as $\tau \to 0$). Note that $q_{m+1}(\tau)$ is expanded using Taylor series as $q_{m+1}(\tau) = -1 + O(\tau)$ [6], where $O(\tau) \to 0$ and $O(\tau)/\tau \to c(\neq 0)$ as $\tau \to 0$.

Lemma 6 *Let $L_\tau(z) = \frac{O(\tau)}{z-q(\tau)}, \frac{O(\tau)}{z^2-q(\tau)}$ or $\frac{1}{z-p(\tau)}$, where $q(\tau) = 1 + O(\tau)$ and $|p(\tau)| \to d(\neq 1)$ as $\tau \to 0$. Then, $\left\|\mathcal{Z}^{-1}\left[L_\tau(z)\mathcal{Z}[v(k)]\right](k)\right\|_\infty < \infty$ for any small $\tau > 0$ and $\|v(k)\|_\infty < \infty$.*

PROOF. See appendix A.

Lemma 7 *$\|\mathcal{Z}^{-1}\left[J_\tau(z)\mathcal{Z}[v(k)]\right]\|_\infty < \infty$ for any small $\tau > 0$ and $\|v(k)\|_\infty < \infty$.*

PROOF. See appendix B.

Lemma 8 *Let $\{a_0, \cdots, a_n\}$ and $\{b_1^n, \cdots, b_n^n\}$ be defined as $(z-1)^n = a_0 z^n + \cdots + a_n$ and $B_n(z) = b_1^n z^{n-1} + \cdots + b_n^n$. Then,*

$$\sum_{l=0}^{n} a_{n-l} l^k = \begin{cases} 0 & \text{for } k = 0, \ldots, n-1 \\ n! & \text{for } k = n \end{cases} \tag{16}$$

and $\sum_{l=1}^{n} b_l^n = n!$ for any positive integer n.

PROOF. See appendix C.

Lemma 9 *Assuming the same condition for u_d as in Theorem 4, we obtain $\|\mathcal{Z}^{-1}[K_\tau(z)\mathcal{Z}[v_\tau(k\tau)]](k)\|_\infty \to 0$ as $\tau \to 0$, where $v_\tau(t) = \mathcal{L}^{-1}[G(s)\mathcal{L}[u_d(t) - u_d(\lfloor t/\tau\rfloor\tau)]](t)$.*

PROOF. Consider a function $w(t)$ and assume that $\frac{d}{dt}w, \cdots, \frac{d^{i-1}}{dt^{i-1}}w$ are continuous and $\frac{d^i}{dt^i}w$ exists. Then, $w(t+l\tau)$ ($l = 0, 1, \cdots$) is expanded in the Taylor series as follows: $w(t+l\tau) = w(t) + l\tau\frac{d}{dt}w(t) + (l\tau)^2\frac{d^2}{dt^2}w(t)/2 \cdots + (l\tau)^i\frac{d^i}{dt^i}w(t+\theta_l^t l\tau)/i!$, where $\theta_l^t \in (0,1)$. From this expansion with $t = k\tau$ and Lemma 8, we have

$$\mathcal{Z}^{-1}\left[\frac{(z-1)^i}{\tau^i}\mathcal{Z}[w(k\tau)]\right](k) = \frac{1}{i!}\sum_{l=0}^{i} a_{i-l} l^i \frac{d^i}{dt^i}w(k\tau + \theta_l^k l\tau) \tag{17}$$

for $k = 0, \pm 1, \cdots$. Since (13) or (15) leads to $K_\tau(z) = \frac{(z-1+O(\tau))^p}{\tau^p} = \frac{(z-1)^p}{\tau^p} + \frac{(z-1)^{p-1}}{\tau^{p-1}}\frac{O(\tau)}{\tau} + \cdots + \frac{O(\tau^p)}{\tau^p}$, where $p = n-m$ or $n-m+1$, Lemma 9 is implied by

$$\left\|\mathcal{Z}^{-1}\left[\frac{(z-1)^i}{\tau^i}\mathcal{Z}[v_\tau(k\tau)]\right](k)\right\|_\infty \to 0 \tag{18}$$

as $\tau \to 0$ for $i = 0$ to $n-m$ or $n-m+1$. Note that the relative degree of $G(s)$ is $n-m$. Then, although $u_d(t) - u_d(\lfloor t/\tau\rfloor\tau)$ is discontinuous, $\frac{d^i}{dt^i}v_\tau(t)$ is continuous on $(-\infty, +\infty)$ for $i = 0, \cdots, n-m-1$ and exists for $i = n-m$ and any $t \in (-\infty, +\infty)$. This implies that (17) holds for $w(t) = v_\tau(t)$ ($i = 0, \cdots, n-m$). Moreover, since $\|u_d(t) - u_d(\lfloor t/\tau\rfloor\tau)\|_\infty \to 0$ and the relative degree of $G(s)$ is $n-m$, we obtain $\left\|\frac{d^i}{dt^i}v_\tau(t)\right\|_\infty \to 0$ ($i =, \cdots, n-m$) as $\tau \to 0$. This fact in conjunction with (17) leads to (18) for $i = 0, \cdots, n-m$. This establishes Lemma 9 for $G(s)$ with an odd relative degree. We will now consider $G(s)$ with an even relative degree. To prove (18) for $i = n-m+1$, we decompose $G(s)$ as $G(s) = \frac{K}{s^{n-m}} + \bar{G}(s)$. Since $\frac{Q_m(s)}{P_n(s)} = \frac{1}{s^{n-m}} - \frac{P_n(s) - s^{n-m}Q_m(s)}{P_n(s)s^{n-m}}$ for the monic polynomials $Q_m(s)$ and $P_n(s)$ with orders m and n, respectively, the relative degree of $\bar{G}(s)$ is equal to or more than $n-m+1$. For $\bar{G}(s)$ we can similarly demonstrate the validity of (18) for $i = n-m+1$. Let $\tilde{v}_\tau(t) = \mathcal{L}^{-1}\left[\frac{1}{s^{n-m}}\mathcal{L}[u_d(t) - u_d(\lfloor t/\tau\rfloor\tau)]\right](t)$. From the definition of $H_\tau(z)$, (17) and Lemma 8, we have

$$\begin{aligned} &\frac{(z-1)^{n-m+1}}{\tau^{n-m+1}}\mathcal{Z}\left[\tilde{v}_\tau(k\tau)\right] \\ =&\frac{(z-1)^{n-m+1}}{\tau^{n-m+1}}\left\{\mathcal{Z}\left[\mathcal{L}^{-1}\left[\frac{1}{s^{n-m}}\mathcal{L}[u_d(t)]\right](k\tau)\right] - \frac{\tau^{n-m}B_{n-m}(z)}{(n-m)!(z-1)^{n-m}}\mathcal{Z}[u_d(k\tau)]\right\} \qquad (19) \\ =&\frac{z-1}{\tau}\left\{\mathcal{Z}\left[\frac{1}{(n-m)!}\sum_{l=0}^{n-m} a_{n-m-l} l^{n-m} u_d(k\tau + \theta_l^k l\tau) - \frac{1}{(n-m)!}\sum_{l=1}^{n-m} b_l^{n-m} u_d(k\tau + (n-m-l)\tau)\right]\right\} \qquad (20) \end{aligned}$$

Moreover, with the mean value theorem, (20) leads to

$$\mathcal{Z}\left[\frac{1}{(n-m)!}\left\{\sum_{l=0}^{n-m} a_{n-m-l}l^{n-m}\frac{d}{dt}u_d(k\tau+\theta_l^k l\tau+\phi_l^k\tau)\right.\right.$$
$$\left.\left.-\sum_{l=1}^{n-m} b_l^{n-m}\frac{d}{dt}u_d(k\tau+(n-m-l)\tau+\psi_l^k\tau)\right\}\right] \quad (21)$$

where $\phi_l^k \in (0, n-m+1)$ and $\psi_l^k \in (0,1)$. Note that $\frac{d}{dt}u_d(k\tau+\theta_l^k l\tau+\phi_l^k\tau) \to \frac{d}{dt}u_d(k\tau)$ and $\frac{d}{dt}u_d(k\tau+(n-m-l)\tau+\psi_l^k\tau) \to \frac{d}{dt}u_d(k\tau)$ uniformly for $t = k\tau$ as $\tau \to 0$. Subsequently, from Lemma 8, we conclude (18) for $v_\tau(t) = \tilde{v}_\tau(t)$. □

Proof of Theorem 4 Lemma 7 and 9 imply (11). This convergence along with the inequality (10) establishes Theorem 4. □

4 Conclusion

In this article, a problem on approximation of the inversion of continuous-time systems by the inversion of sampled-data systems with a piecewise constant input was posed. Given was a positive conclusion to the problem under an assumption of smoothness of trajectories in the noncausal framework. The result guarantees that inversion of sampled-data systems can be substituted for the continuous-time counterpart in control applications such as perfect tracking, transient response shaping or iterative learning control. It should be noted that the result holds not only for continuous-time systems with no unstable zero such as the one given in Example 1 but also for continuous-time systems with unstable zeros. In such cases, it is observed that noncausal piecewise constant inputs approximate a noncausal continuous input that achieves the desired trajectory.

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A Proof of Lemma 6

From the definition of the two-sided z-transform, we have

$$w(k) = \mathcal{Z}^{-1}\left[\frac{1}{z-r(\tau)}\mathcal{Z}[v(l)]\right] \quad (A.1)$$

$$= \begin{cases} \sum_{l=-\infty}^{k} r(\tau)^{k-l}v(l) & \text{if } |r(\tau)|<1 \\ \sum_{l=-k+1}^{+\infty} r(\tau)^{k-l}v(l) & \text{if } |r(\tau)|>1 \end{cases} \quad (A.2)$$

This leads to

$$|w(k)| \le \begin{cases} \frac{1}{1-r(\tau)}\|v(l)\|_\infty & \text{if } |r(\tau)| < 1 \\ \frac{1}{r(\tau)-1}\|v(l)\|_\infty & \text{if } |r(\tau)| > 1 \end{cases} \tag{A.3}$$

for any k. A similar formula for $\frac{1}{z^2-r(\tau)}$ holds likewise. Lemma 6 follows from the above inequalities. □

B Proof of Lemma 7

From the definition of $H_\tau(z)$, we have $cb_\tau = c(\tau^{n-m}A^{n-m+1}/(n-m)! + \tau^{n-m+1}A^{n-m+2}/(n-m+1)! + \cdots)b$. This implies that $|\tau^{n-m}/(cb_\tau)| < \infty$ for any small $\tau > 0$. Since $q_i(\tau) = 1 + \gamma_i\tau + O(\tau^2)$ for $i = 1, \cdots, m$, we have $\frac{(z-\exp(p_1\tau))\cdots(z-\exp(p_m\tau))}{(z-q_1(\tau))\cdots(z-q_m(\tau))} = 1 + \sum_i \frac{O(\tau)}{z-q_i(\tau)} + \sum_{i,j}\frac{O(\tau)}{z-q_i(\tau)}\frac{O(\tau)}{z-q_j(\tau)} + \cdots$, which shows that $J_\tau(z)$ consists of linear combination and product of $\frac{O(\tau)}{z-q(\tau)}$, $\frac{O(\tau)}{z^2-q(\tau)}$ and $\frac{1}{z-p(\tau)}$ with bounded scalars. This relation and Lemma 6 imply Lemma 7. □

C Proof of Lemma 8

Consider a polynomial $p_n(t) = c_0t^n + \cdots + c_{n-1}t + c_n$ and the nth divided difference $p_n[i\tau, \cdots, (i+n)\tau]$ of $p_n(t)$, which is recursively defined as $p_n[i\tau, \cdots, (i+n)\tau] = \frac{p_n[(i+1)\tau, \cdots, (i+n)\tau] - p_n[i\tau, \cdots, (i+n-1)\tau]}{(i+n)\tau - i\tau}$ and $p_n[i\tau] = p_n(i\tau)$ $(i = 0, \pm 1, \cdots, \pm n)$[2]. Since the nth divided difference is equal to the leading coefficient of the polynomial interpolating at the points $\{i\tau, \cdots, (i+n)\tau\}$[2], we have

$$p_n[i\tau, \cdots, (i+n)\tau] = c_0 \tag{C.1}$$

It should be noted that (16) is expressed as follows using the nth divided difference of $q_k(t) = t^k$:

$$q_k[0, \tau, \cdots, n\tau] = \begin{cases} 0 & \text{for } k = 0, \ldots, n-1 \\ 1 & \text{for } k = n \end{cases} \tag{C.2}$$

First, (C.1) directly leads to the case $k = n$ in (C.2) for any positive n. It is evident that the case $k = 0, \ldots, n-1$ in (C.2) holds for $n = 1$. Assume that this case in (C.2) holds for $n = m$. Then, it follows from the definition of the divided difference and (C.1) that (C.2) holds for $n = m + 1$. By induction, we establish (16) for any positive integer n.

Note that the discrete-time transfer function of n-tuple integrators $G(s) = s^{-n}$ with a piecewise constant function is expressed as $\tau^n B_n(z)/\{n!(z-1)^n\}$, which is defined using sampled data of the step response of $G(s) = s^{-n}$, namely, $y(t) = t^n/n!$ [1]. Then, we have $\sum_{l=0}^n a_{n-l}\frac{l^n\tau^n}{n!} = \frac{\tau^n}{n!}\sum_{l=1}^n b_l^n$ This formula and (16) with $k = n$ imply $\sum_{l=1}^n b_l^n = n!$. □