Simple algebraic structure of Taylor expansion of sampled-data systems

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***Abstract*—The relation between the continuous-time model and the corresponding discrete-time model of sampled-data system has not been believed to be very simple. However, from the view point of Taylor expansion with respect to sample time, the relation is approximated by unexpectedly simple polynomials. In this paper, we show that there is a simple regularity in Taylor expansion for any sampled-data systems. Next, it is demonstrated that the regularity reduces symbolic calculation of the Taylor expansion. Finally, we apply the result to identification of continuous-time model from discrete-time input-output data of sampled-data systems based on optimization techniques.**

I. Introduction

Linear dynamic system theory has been exhaustively developped for both continuous-time and discrete-time systems. Since most controllers for recent industrial applications are implemented as digital computer systems, discretization by sample and hold operations is an indispensable part of the control systems. This implies that the interesting discrete-time systems are necessarily related with the continuous-time systems. However, their relationship is not very simple. Let's consider a single-input-single-output linear time-invariant system (A_c, B_c, C) with the transfer function

$$G(s) = C(sI - A_c)^{-1}B_c \tag{1}$$

$$= \frac{b_0 s^m + b_1 s^{m-1} + \cdots + b_m}{s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n} \tag{2}$$

$$= \frac{K(s-q_1)\cdots(s-q_m)}{(s-p_1)(s-p_2)\cdots(s-p_n)} \tag{3}$$

Then the discrete-time system generated by sampler and zero-order hold of a sample period τ is written by the system matrices (A, B, C) where

$$A = \exp(A_c\tau) = \sum_{k=0}^{\infty} \frac{\tau^k}{k!} A_c^k \tag{4}$$

$$B = \int_0^\tau \exp(A_c t)\, dt B_c = \sum_{k=0}^{\infty} \frac{\tau^{k+1}}{(k+1)!} A_c^k B_c \tag{5}$$

and the pulse transfer function

$$H(z) = C(zI - A)^{-1}B \tag{6}$$

$$= \frac{\begin{vmatrix} A - zI & B \\ C & 0 \end{vmatrix}}{|A - zI|} \tag{7}$$

$$= \frac{\beta_1(\tau)z^{n-1} + \beta_2(\tau)z^{n-2} + \cdots + \beta_n(\tau)}{z^n + \alpha_1(\tau)z^{n-1} + \cdots + \alpha_{n-1}(\tau)z + \alpha_n(\tau)} \tag{8}$$

Moreover, from the expression

$$\frac{G(s)}{s} = \frac{r_0}{s} + \frac{r_1}{s-p_1} + \cdots + \frac{r_n}{s-p_n} \tag{9}$$

we derive another expression of the pulse transfer function

$$H(z) = \frac{z-1}{z}\left\{\frac{r_0 z}{z - e^0} + \frac{r_1 z}{z - e^{p_1\tau}} + \cdots + \frac{r_n z}{z - e^{p_n\tau}}\right\} \tag{10}$$

$$= \frac{C_\tau\{z - \gamma_1(\tau)\}\cdots\{z - \gamma_{n-1}(\tau)\}}{(z - e^{p_1\tau})(z - e^{p_2\tau})\cdots(z - e^{p_n\tau})} \tag{11}$$

It is well-known that the discrete-time system has $n - m - 1$ so-called discretization zeros, which are often unstable even though the continuous-time system has no unstable zero. This hinders efficient design of discrete-time controller for continuous-time systems. No simple relation other than the poles is expected between the continuous-time and discrete-time systems. However, calculating Taylor expansion with respect to the sample period τ, we can observe that there is a simple regularity in the terms of the coefficients of both the denominator and numerator. To illustrate the regularity, let's consider the transfer function (3) for the case where $(n, m) = (3, 1)$. From the expressions (4), (5) and (7), we can easily calculate the Taylor expansions of the coefficients in both denominator and numerator of the transfer function (8):

$$\alpha_1(\tau) = 3 - \tau a_1 + \frac{\tau^2}{2}(a_1^2 - 2a_2) + \frac{\tau^3}{3!}(-a_1^3 + 3a_1a_2 - 3a_3) + \cdots \tag{12}$$

$$\alpha_2(\tau) = -3 + \tau(2a_1) + \frac{\tau^2}{2}(-2a_1^2 + a_2) + \frac{\tau^3}{3!}(2a_1^3 - 3a_1a_2 - 3a_3) + \cdots \tag{13}$$

$$\alpha_3(\tau) = 1 - \tau a_1 + \frac{\tau^2}{2}a_1^2 + \frac{\tau^3}{3!}a_1^3 + \cdots \tag{14}$$

$$\beta_1(\tau) = \frac{\tau^2}{2} b_0 + \frac{\tau^3}{3!}(-a_1 b_0 + b_1) + \frac{\tau^4}{4!}(a_1^2 b_0 - a_2 b_0 - a_1 b_1) + \cdots \tag{15}$$

$$\beta_2(\tau) = \frac{\tau^2}{2} \times 0 + \frac{\tau^3}{3!}(-a_1 b_0 + 4b_1) + \frac{\tau^4}{4!}(2a_1^2 b_0 - 8a_1 b_1) + \cdots \tag{16}$$

$$\beta_3(\tau) = -\frac{\tau^2}{2} b_0 + \frac{\tau^3}{3!}(2a_1 b_0 + b_1) + \frac{\tau^4}{4!}(-3a_1^2 b_0 + a_2 b_0 - 3a_1 b_1) + \cdots \tag{17}$$

From those expressions, we observe that the index and exponent of each term is restricted by the value of the corresponding exponent of τ. For example, the terms of $\tau^3/3!$ in $\alpha_1(\tau)$ are

$$-a_1^3 + 3a_1 a_2 - 3a_3 \tag{18}$$

where each summation of the indices multiplied by the corresponding exponents is equal to the exponent of τ^3, namely

$$1 \times 3 = 1 \times 1 + 2 \times 1 = 3 \times 1 = 3 \tag{19}$$

This algebraic formula holds for all other terms in the denominator's coefficient $\alpha_1(\tau)$, $\alpha_2(\tau)$ and $\alpha_3(\tau)$. Moreover, we observe that a similar algebraic formula holds for the numerator's coefficients $\beta_1(\tau)$, $\beta_2(\tau)$ and $\beta_3(\tau)$. For example, the terms of $\tau^4/4!$ in $\beta_1(\tau)$ are

$$a_1^2 b_0 - a_2 b_0 - a_1 b_1 \tag{20}$$

where each summation of the indices multiplied by the corresponding exponents is equal to the exponent of τ^4 decreased by the relative degree $n - m = 2$, namely

$$1 \times 2 + 0 \times 1 = 2 \times 1 + 0 \times 1 = 1 \times 1 + 1 \times 1 = 4 - (n - m) = 2 \tag{21}$$

In addition, this formula implies that the terms of τ^0 and τ^1 vanish because any non-negative index and exponent cannot let the summations be $0 - (n - m) = -2$ and $1 - (n - m) = -1$, respectively. This algebraic formula holds for all terms in $\beta_1(\tau)$, $\beta_2(\tau)$ and $\beta_3(\tau)$. Thanks to those algebraic regularities, all parameters in the discrete-time systems are approximately expressed by relatively simple polynomials of the parameters of the continuous-time systems. This fact helps us identify the continuous-time model from the sampled data and design sophisticated discrete-time controllers.

In this paper, we show that any linear systems satisfy the above mentioned regularities from the viewpoint of multivariable algebra. Next, we demonstrate that the regularities reduce the calculation amount for the Taylor expansion of the coefficients. Finally, we develop an approach to system identification of the continuous-time system from the discrete-time data.

II. Main results

Theorem 1. *The Taylor expansion of every denominator's coefficient of $H(z)$ is written by*

$$\alpha_i(\tau) = \sum_{k=0}^{\infty} \frac{\tau^k}{k!} \left(\sum_{\boldsymbol{\nu} \in \boldsymbol{N}_0^n} c(k, \boldsymbol{\nu}) \cdot a_1^{\nu_1} a_2^{\nu_2} \cdots a_n^{\nu_n} \right) \tag{22}$$

where $\boldsymbol{\nu} = (\nu_1, \cdots, \nu_n) \in \boldsymbol{N}_0^n$ are n non-negative integers and $c(k, \boldsymbol{\nu})$'s are real constants satisfying

$$c(k, \boldsymbol{\nu}) = 0 \tag{23}$$

if

$$k \neq 1 \cdot \nu_1 + 2 \cdot \nu_2 + \cdots + n \cdot \nu_n \tag{24}$$

Remark 1. *The summation $\sum_{\boldsymbol{\nu} \in \boldsymbol{N}_0^n}$ in the equation (22) with the condition (23) and (24) is essentially reduced to the finite summation only for the combination $\boldsymbol{\nu} = (\nu_1, \cdots, \nu_n)$ satisfying $k = 1 \cdot \nu_1 + 2 \cdot \nu_2 + \cdots + n \cdot \nu_n$. Moreover, this implies that the parameters $a_{k+1}, \cdots, a_n$ do not emerge in the term of τ^k when k is less than n. For example, when $k = 1, 2$ or 3, the possible combinations of the exponents are:*

k	$(\nu_1, \nu_2, \cdots, \nu_n)$	
1	$(1, 0, \cdots, 0)$	a_1
2	$(2, 0, \cdots, 0)$	a_1^2
	$(0, 1, \cdots, 0)$	a_2
3	$(3, 0, 0, \cdots, 0)$	a_1^3
	$(1, 1, 0, \cdots, 0)$	$a_1 a_2$
	$(0, 0, 1, \cdots, 0)$	a_3

Theorem 2. *The Taylor expansion of every numerator's coefficient of $H(z)$ is written by*

$$\beta_i(\tau) = \sum_{k=0}^{\infty} \frac{\tau^k}{k!} \left(\sum_{\boldsymbol{\mu} \in \boldsymbol{N}_0^n, j \in \boldsymbol{N}_0} c(k, \boldsymbol{\mu}, j) \cdot a_1^{\mu_1} a_2^{\mu_2} \cdots a_n^{\mu_n} b_j \right) \tag{25}$$

where $\boldsymbol{\mu} = (\mu_1, \cdots, \mu_n) \in \boldsymbol{N}_0^n$ are n non-negative integers and $c(k, \boldsymbol{\mu}, j)$'s are real constants satisfying

$$c(k, \boldsymbol{\mu}, j) = 0 \tag{26}$$

if

$$k - (n - m) \neq 1 \cdot \mu_1 + 2 \cdot \mu_2 + \cdots + n \cdot \mu_n + j \tag{27}$$

Remark 2. *Similarly as Remark 1, the summation $\sum_{\boldsymbol{\mu} \in \boldsymbol{N}_0^n, j \in \boldsymbol{N}_0}$ in the equation (25) is essentially reduced to the finite summation only for the combination implied by the condition (26) with (27). Moreover, when the relative degree $n - m$ is positive, there exist no combination for the exponents $(\mu_1, \cdots, \mu_n)$ and the index j for small k because they are non-negative. For example, when $n - m = 2$, there is no combination of $(\mu_1, \cdots, \mu_n)$ and j that satisfies $k - 2 = 1 \cdot \mu_1 + 2 \cdot \mu_2 + \cdots + n \cdot \mu_n + j$ for $k = 0$ or 1, which means that the terms of τ^0 and τ^1 vanish and the Taylor expansion (25) begins from the term of $\tau^2/2!$. This property is consistent with the known fact*

$$\frac{H(z)}{\tau^{n-m}} \to \frac{(z-1)^m Q_{n-m-1}(z)}{(z-1)^n} \tag{28}$$

as $\tau \to 0$ where $Q_{n-m-1}(z)$ is the Euler-Frobenius polynomial[1], [4].

As shown in the previous section, every example of Theorem 1 and 2 is directly demonstrated by caluculating the equation (7). However, Theorem 1 and 2 are mathematically proved from the equations (10) and (11) by using the relation between the coefficients and the poles and zeros, namely

(a_i, b_j) and (p_i, q_j) $(i = 1, \cdots, n; j = 0, \cdots, m)$ as described below in Remark 3. To do so, we prepare some technical terms for multi-variable polynomial algebra, which seems less common than single-variable polynomial algebra. We regard poles $p_1, \cdots, p_n$ or zeros $q_1, \cdots, q_m$ as variables of multi-variable polynomial rings below.

Definition 1 (Monomial, Polynomial). *[2] A* **monomial** *in $p_1, \cdots, p_n$ is a product of the form*

$$p_1^{\nu_1} \cdot p_2^{\nu_2} \cdots p_n^{\nu_n} \tag{29}$$

where all of the exponents $\nu_1, \cdots, \nu_n$ are non-negative integers. A **polynomial** *f in $p_1, \cdots, p_n$ with coefficients in R is a finite linear combination of monomials, namely*

$$f(p_1, \cdots, p_n) = \sum_{(\nu_1, \cdots, \nu_n)} \kappa_{(\nu_1, \cdots, \nu_n)} p_1^{\nu_1} \cdot p_2^{\nu_2} \cdots p_n^{\nu_n}, \tag{30}$$

where $\kappa_{(p_1, \cdots, p_n)} \in R$ and the sum is over a finite number of n-tuples $(\nu_1, \cdots, \nu_n)$. The set of all polynomials in $p_1, \cdots, p_n$ with coefficients in R is denoted by $R[p_1, \cdots, p_n]$. The maximum value of the sum $\nu_1 + \cdots + \nu_n$ for non-zero coefficient $\kappa_{(\nu_1, \cdots, \nu_n)}$ of a polynomial f is called the **total degree** *of f and is denoted as $\overline{\deg}(f)$. In this paper, when all the sum $\nu_1 + \cdots + \nu_n$ for non-zero coefficient $\kappa_{(\nu_1, \cdots, \nu_n)}$ is equal each other, the sum is called just as* **degree** *of the polynomial f and denoted as* $\deg(f)$.

Definition 2 (Symmetric polynomial). *A polynomial $S \in R[p_1, \cdots, p_n]$ is called as* **symmetric** *if*

$$S(p_1, \cdots, p_n) = S(p_{\sigma(1)}, \cdots, p_{\sigma(n)}) \tag{31}$$

for any permutation σ of the subscripts $1, \cdots, n$.

Definition 3 (Elementary symmetric polynomial). *The elementary symmetric polynomials $e_{n,1}, \cdots, e_{n,n} \in R[p_1, \cdots, p_n]$ are defined as*

$$e_{n,1} = p_1 + \cdots + p_n \tag{32}$$

$$\vdots \tag{33}$$

$$e_{n,j} = \sum_{1 \le i_1 \le i_2 \cdots \le i_j \le n} p_{i_1} p_{i_2} \cdots p_{i_j} \tag{34}$$

$$\vdots \tag{35}$$

$$e_{n,n} = p_1 p_2 \cdots p_n. \tag{36}$$

In this paper, we define $e_{n,0} = 1$ as one of the elementary symmetric polynomials, which will simplify the following proof.

Remark 3. *The coefficients in the transfer function (2) are related with the elementary symmetric polynomials of the poles or zeros, i.e.*

$$b_j / b_0 = (-1)^j e_{m,j}(q_1, \cdots, q_m) \quad (j = 1, \cdots, m) \tag{37}$$

$$a_i = (-1)^i e_{n,i}(p_1, \cdots, p_n) \quad (i = 1, \cdots, n). \tag{38}$$

Moreover, $\deg(b_0) = 0$*, and*

$$\deg(b_j) = \deg(b_j/b_0) = \deg(e_{m,j}(q_1, \cdots, q_m)) = j \tag{39}$$

$$\deg(a_i) = \deg(e_{n,i}(p_1, \cdots, p_n)) = i \tag{40}$$

with respect to $\{q_1, \cdots, q_m\}$ or $\{p_1, \cdots, p_n\}$.

Theorem 1 is directly proved by using the following proposition known as the fundamental theorem of symmetric polynomials[2]:

Proposition 1. *Any symmetric polynomial $f \in R[p_1, \cdots, p_n]$ is uniquely expressed by the elementary symmetric polynomials $e_{n,i} \in R[p_1, \cdots, p_n]$*

$$f = \sum_{(\mu_1, \cdots, \mu_n)} \kappa_{(\mu_1, \cdots, \mu_n)} e_{n,1}^{\mu_1} \cdots e_{n,n}^{\mu_n} \tag{41}$$

Corollary 1. *If a symmetric polynomial f consists of monomials with an identical degree, the polynomial f denoted as (41) satisfies*

$$\deg(f) = 1 \cdot \mu_1 + 2 \cdot \mu_2 + \cdots + n \cdot \mu_n. \tag{42}$$

Proof of Theorem 1: From (11), we express the denominator of the transfer function $H(z)$ as

$$\begin{aligned} & z^n - (e^{p_1 \tau} + \cdots + e^{p_n \tau}) z^{n-1} \\ & + (e^{(p_1+p_2)\tau} + \cdots + e^{(p_{n-1}+p_n)\tau}) z^{n-2} \\ & + \cdots + (-1)^n e^{(p_1 + \cdots + p_n)\tau} \end{aligned} \tag{43}$$

$$\begin{aligned} =& z^n - z^{n-1} \sum_{k=0}^{\infty} (p_1^k + \cdots + p_n^k) \tau^k / k! \\ & + z^{n-2} \sum_{k=0}^{\infty} \{(p_1 + p_2)^k + \cdots + (p_{n-1} + p_n)^k\} \tau^k / k! \\ & + \cdots + (-1)^n \sum_{k=0}^{\infty} (p_1 + \cdots + p_n)^k \tau^k / k! \end{aligned} \tag{44}$$

$$\begin{aligned} =& z^n - z^{n-1} \sum_{k=0}^{\infty} S_{(1,k)}(p_1, \cdots, p_n) \tau^k / k! \\ & + z^{n-2} \sum_{k=0}^{\infty} S_{(2,k)}(p_1, \cdots, p_n) \tau^k / k! \\ & + \cdots + (-1)^n \sum_{k=0}^{\infty} S_{(n,k)}(p_1, \cdots, p_n) \tau^k / k! \end{aligned} \tag{45}$$

where $S_{(i,k)}(p_1, \cdots, p_n)$'s $(i = 1, \cdots, n)$ are symmetric polynomials satisfying $\deg(S_{(i,k)}) = k$. By using Proposition 1 with (38) and Corollary 1 with (40), we conclude Theorem 1 from (45). ■

Unlike Theorem 1, Theorem 2 is not directly concluded from Proposition 1 or Corollary 1. We prepare the following Lemma. For simplicity of discussions, we define $p_0 = 0$ as one of the poles of (9) and consider polynomial of the poles $\{p_0, p_1, \cdots, p_n\}$ with the residuals $\{r_0, r_1, \cdots, r_n\}$ of (9).

Lemma 1. *Consider a polynomial*

$$\sum_{\mu=0}^{n} r_\mu S(\{p_0, p_1, \cdots, p_n\} \backslash \{p_\mu\}) \tag{46}$$

where S is any n-variable symmetric polynomial and $\{p_0, p_1, \cdots, p_n\} \backslash \{p_\mu\}$ denotes the set of poles $\{p_0, p_1, \cdots, p_n\}$ except $\{p_\mu\}$. This polynomial can be divided by the set of $n+1$ polynomials

$$\sum_{\mu=0}^{n} r_\mu e_{n,k}(\{p_0, p_1, \cdots, p_n\} \backslash \{p_\mu\}) \tag{47}$$

$(k=0,1,\cdots,n)$ and written by

$$\sum_{k=0}^{n}\left[X_k\cdot\sum_{\mu=0}^{n}r_\mu e_{n,k}(\{p_0,p_1,\cdots,p_n\}\backslash\{p_\mu\})\right] \tag{48}$$

where X_k's are $n+1$-variable polynomial in $e_{n+1,i}$ $(i=1,\cdots,n+1)$ that is the elementary symmetric polynomial in $n+1$ variables $\{p_0,p_1,\cdots,p_n\}$. Moreover, if $\deg(S)$ is defined, it satisfies

$$\deg(S)=\deg(X_k)+k. \tag{49}$$

Outline of proof: Lemma 1 is proved by mathematical induction with respect to the number of variables $\{p_0,p_1,\cdots,p_n\}$ with $\{r_0,r_1,\cdots,r_n\}$.

We consider the elementary symmetric polynomials of the variables $\{p_0,p_1,\cdots,p_n\}$, namely $e_{n+1,k}(p_0,p_1,\cdots,p_n)$. From the definition, we get the following recurrence formulas:

$$e_{n+1,1}(p_0,p_1,\cdots,p_n)=e_{n,1}(p_0,\cdots,p_{n-1})+p_n \tag{50}$$

$$\begin{aligned}e_{n+1,2}(p_0,p_1,\cdots,p_n)=&e_{n,2}(p_0,\cdots,p_{n-1})\\&+e_{n,1}(p_0,\cdots,p_{n-1})p_n\end{aligned} \tag{51}$$

$$\vdots$$

$$\begin{aligned}e_{n+1,k}(p_0,p_1,\cdots,p_n)=&e_{n,k}(p_0,\cdots,p_{n-1})\\&+e_{n,k-1}(p_0,\cdots,p_{n-1})p_n\end{aligned} \tag{52}$$

$$\vdots$$

$$e_{n+1,n+1}(p_0,p_1,\cdots,p_n)=e_{n,n}(p_0,\cdots,p_{n-1})p_n \tag{53}$$

By repetitive substitution for (50), (51), (52) and (53), we obtain

$$\begin{aligned}&(-p_n)^k+(-p_n)^{k-1}e_{n+1,1}+\cdots\\&\quad+(-p_n)e_{n+1,i-1}+e_{n+1,k}=\begin{cases}e_{n,k} & \text{for } k=1,\cdots,n\\ 0 & \text{for } k=n+1\end{cases}\end{aligned} \tag{54}$$

We denote $n+1$ polynomials (47) as $E_{n,k}$ $(k=0,\cdots,n)$ and get the recurrence formulas:

$$E_{n,0}=E_{n-1,0}+r_n \tag{55}$$

$$E_{n,k}=E_{n-1,k}+E_{n-1,k-1}p_n+r_ne_{n,k}\quad(k=1,\cdots,n-1) \tag{56}$$

$$E_{n,n}=E_{n-1,n-1}p_n+r_ne_{n,n} \tag{57}$$

which leads to

$$\begin{aligned}&(-p_n)^kE_{n,0}+(-p_n)^{k-1}E_{n,1}+\cdots+(-p_n)e_{n,k-1}+E_{n,k}\\&-r_n\left\{(k+1)(-p_n)^k+k(-p_n)^{k-1}e_{n+1,1}+\right.\\&\left.\cdots+2(-p_n)e_{n+1,k-1}+e_{n+1,k}\right\}\\&=\begin{cases}E_{n-1,k} & \text{for } k=1,\cdots,n-1\\ 0 & k=n\end{cases}\end{aligned} \tag{58}$$

Next, we express the symmetric polynomial S in n variables as the power series of the variable p_n, namely

$$\begin{aligned}&S(\{p_0,p_1,\cdots,p_n\}\backslash\{p_\mu\})\\&=\sum_{k\geq 0}p_n^kS_k(\{p_0,p_1,\cdots,p_{n-1}\}\backslash\{p_\mu\})\end{aligned} \tag{59}$$

for $\mu=0,\cdots n-1$ where S_k are still symmetric polynomials in $n-1$ variables. By these expressions, Proposition 1 and the assumption of mathematical induction, the polynomial (46) is written by

$$\begin{aligned}&\sum_{k\geq 0}p_n^k\sum_{\mu=0}^{n-1}r_\mu S(\{p_0,p_1,\cdots,p_{n-1}\}\backslash\{p_\mu\})\\&\quad+r_nS(\{p_0,p_1,\cdots,p_{n-1}\})\\&=\sum_{k\geq 0}p_n^k\sum_{i=0}^{n-1}X_{k,i}E_{n,i}+r_nF(e_{n,1},\cdots,e_{n,n})\end{aligned} \tag{60}$$

Substituting (54) and (58) into (60) and reducing the exponent of p_n, we conclude the proof. ■

Here we note that comparing the numerator of the expressions (2) and (9) leads to the equations

$$\begin{aligned}&\sum_{\mu=0}^{n}r_\mu e_{n,k}(\{p_0,\cdots,p_n\}\backslash\{p_\mu\})\\&=\begin{cases}0 & (k=0,\cdots,n-m-1)\\ (-1)^kb_{k-(n-m)} & (k=n-m,\cdots,n)\end{cases}\end{aligned} \tag{61}$$

Moreover, from (39) we have

Proof of Theorem 2: From the expression (10), the numerator of $H(z)$ is written by

$$\sum_{\mu=0}^{n}r_\mu\prod_{k\in\{0,1,\cdots,n\}\backslash\{\mu\}}(z-e^{p_k\tau}) \tag{62}$$

$$\begin{aligned}&=z^n\sum_{\mu=0}^{n}r_\mu-z^{n-1}\sum_{\mu=0}^{n}r_\mu\sum_{l=0}^{\infty}\frac{\tau^l}{l!}S_{(1,l)}(\{p_0,\cdots,p_n\}\backslash\{p_\mu\})\\&\quad+z^{n-2}\sum_{\mu=0}^{n}r_\mu\sum_{l=0}^{\infty}\frac{\tau^l}{l!}S_{(2,l)}(\{p_0,\cdots,p_n\}\backslash\{p_\mu\})\\&\quad+\cdots+(-1)^n\sum_{\mu=0}^{n}r_\mu\sum_{l=0}^{\infty}\frac{\tau^l}{l!}S_{(n,l)}(\{p_0,\cdots,p_n\}\backslash\{p_\mu\})\end{aligned} \tag{63}$$

where $S_{(i,l)}$'s are symmetric polynomial

$$S_{(i,l)}(x_1,\cdots,x_n)=\sum_{\substack{\text{for all combinations of}\\ i\text{ numbers }\{j_1,\cdots,j_i\}}}(x_{j_1}+\cdots+x_{j_i})^l \tag{64}$$

We note that $\deg(S_{(i,l)})=l$ for $i=1,\cdots,n$. Applying Lemma 1, we express the coefficient of z^j of (63) as

$$(-1)^j\sum_{\mu=0}^{n}r_\mu\sum_{l=0}^{\infty}\frac{\tau^l}{l!}S_{(j,l)}(\{p_0,\cdots,p_n\}\backslash\{p_\mu\})$$

$$=(-1)^j\sum_{l=0}^{\infty}\frac{\tau^l}{l!}\sum_{\mu=0}^{n}r_\mu S_{(j,l)}(\{p_0,\cdots,p_n\}\backslash\{p_\mu\}) \tag{65}$$

$$=(-1)^j\sum_{l=0}^{\infty}\frac{\tau^l}{l!}\sum_{k=0}^{n}\left[X_k\cdot\sum_{\mu=0}^{n}r_\mu e_{n,k}(\{p_0,p_1,\cdots,p_n\}\backslash\{p_\mu\})\right] \tag{66}$$

where

$$l=\deg(X_k)+k \tag{67}$$

Since $p_0 = 0$, we have $e_{n+1,i}(p_0, p_1, \cdots, p_n) = e_{n,i}(p_1, \cdots, p_n)$ and

$$X_k = \sum_{(\nu_1,\cdots,\nu_n)} \kappa_{(\nu_1,\cdots,\nu_n)} e_{n,1}^{\nu_1} \cdots e_{n,n}^{\nu_n} \quad (68)$$

$$= \sum_{(\nu_1,\cdots,\nu_n)} \kappa_{(\nu_1,\cdots,\nu_n)} (-1)^1 a_1^{\nu_1} \cdots (-1)^n a_n^{\nu_n} \quad (69)$$

The equations (61), (66) and (69) with (67) conclude Theorem 2. ■

III. Applications

A. Reduction of symbolic calculation for Taylor expansion

We can calculate Taylor expansion of $\alpha_i(\tau)$ or $\beta_i(\tau)$ with respect to τ by calculation of the matrix determinant (7) with the truncated sum of the series (4) and (5). Since the calculation consists only of multiplication and addition, it is easily aided by software tools for symbolic mathematical calculation, e.g. Maple. For example, we consider the case of $(n, m) = (8, 5)$ and calculate Taylor expansion of $\beta_2(\tau)$ up to the 5th-order term. To do so, it is sufficient to let

$$A_c = \begin{bmatrix} 0 & & & \\ \vdots & & I & \\ 0 & & & \\ -a_8 & \cdots & \cdots & -a_1 \end{bmatrix} \quad (70)$$

$$B_c = \begin{bmatrix} 0 & \cdots & 0 & 1 \end{bmatrix}^T \quad (71)$$

$$C = \begin{bmatrix} b_5 & \cdots & b_0 & 0 & 0 \end{bmatrix} \quad (72)$$

$$A = I + A_c\tau + \cdots, A_c^5\tau^5/5! \quad (73)$$

$$B = B_c\tau + A_cB_c\tau^2/2! + \cdots + A_c^4B_c\tau^5/5! \quad (74)$$

and calculate the determinant in the numerator of (7). Then we have

$$\begin{aligned}\beta_2 = &- b_0\tau^3/3! + (7b_1 - 3a_1b_0)\tau^4/4! \\ &+ (-3a_2b_0 + 8a_1^2b_0 - 18a_1b_1 + 23b_2)\tau^5/5! \\ &+ P_6 \cdot \tau^6/6! + P_7 \cdot \tau^7/7! + \cdots + P_{5\times 9} \cdot \tau^{5\times 9}/(5 \times 9)!\end{aligned} \quad (75)$$

where P_k's ($k = 6, 7, \cdots, 5 \times 9$) are polynomials in $\{a_1, \cdots, a_8, b_0, b_1, \cdots, b_5\}$. We obtain the correct Taylor expansion up to the 5th-order term in (75). On the other hand, the polynomials P_k's which consist of large number of monomials are meaningless and wast computational resources because the matrices (73) and (74) are truncated up to the 5th-order term. However, since Theorem 2 implies that the coefficient a_i and b_i with the subscript i more than $5 - (n - m) = 2$ never emerge in the Taylor expansion up to the 5th-order term, we can reduce the calculation of the determinant in the numerator of (7) by just letting $a_i = 0$ and $b_i = 0$ for $i > 3$. The efficacy of this technique is shown in Table I.

The calculation of the Taylor expansion based on the matrix determinant in the equation (7) is essentially inefficient because it is difficult to reduce all of the meaningless higer terms with respect to τ even though the theoretically useless coeffients are eliminated as demonstrated above. (e.g. many greater-order terms than τ^5 are still generated in the equation (75)) However, based on the approach of the proof for Theorem 1 and 2 and Lemma 1, we can directly calculate the interesting-order term without generating any useless terms. To demonstrate the approach, we consider the same example of $(n, m) = (8, 5)$. From the equation (10), we express

	P_6	P_7	P_8	P_9	P_{10}	Total
Monomials without reduction	7	11	18	22	36	94
Monomials with reduction	5	6	7	5	5	28

TABLE I. The numbers of monomials with or without reduction for calculating β_2

$$\begin{aligned}\beta_2 = &\left\{e^{(p_1+p_2)\tau} + \cdots + e^{(p_7+p_8)\tau}\right\} r_0 + \cdots \\ &+ \left\{e^{(p_0+p_1)\tau} + \cdots + e^{(p_6+p_7)\tau}\right\} r_8\end{aligned} \quad (76)$$

$$\begin{aligned}= &\sum_{k=0}^{\infty} \frac{\tau^k}{k!} \left[\left\{(p_1 + p_2)^k + \cdots + (p_7 + p_8)^k\right\} r_0 + \cdots \right. \\ &\left. + \left\{(p_0 + p_1)^k + \cdots + (p_6 + p_7)^k\right\} r_8\right]\end{aligned} \quad (77)$$

where $p_0 = 0$. On the other hand, by comparing the equations (2) and (9) we have

$$0 = r_0 + r_1 + \cdots + r_8 \quad (78)$$

$$\begin{aligned}0 = &r_0(p_1 + \cdots + p_8) + r_1(p_0 + p_2 + \cdots + p_8) + \cdots \\ &+ r_8(p_0 + \cdots + p_7)\end{aligned} \quad (79)$$

$$0 = r_0(p_1p_2 + \cdots + p_7p_8) + \cdots + r_8(p_0p_1 + \cdots + p_6p_7) \quad (80)$$

$$\begin{aligned}b_0 = &r_0(p_1p_2p_3 + \cdots + p_6p_7p_8) + \cdots \\ &+ r_8(p_0p_1p_2 + \cdots + p_5p_6p_7)\end{aligned} \quad (81)$$

$$\vdots$$

$$b_5 = r_0p_1 \cdots p_8 + r_1p_0p_2 \cdots p_8 + \cdots + r_8p_0 \cdots p_7 \quad (82)$$

By using the procedures of the proof for Theorem 2 and Lemma 1, each term in the summation (77) for k is expressed by the left-hand sides (78), $\cdots$ (82) and (38) which are equivalent to the coefficients $b_j (j = 0, \cdots, m)$ and a_i $(i = 1, \cdots, n)$. For example, we directly calculate

$$\begin{aligned}&\tau^5/5! \left[\left\{(p_1 + p_2)^5 + \cdots + (p_7 + p_8)^5\right\} r_0 + \cdots \right. \\ &\left. \cdots + \left\{(p_0 + p_1)^5 + \cdots + (p_6 + p_7)^5\right\} r_8\right] \\ = &(-3a_2b_0 + 8a_1^2b_0 - 18a_1b_1 + 23b_2)\tau^5/5!\end{aligned} \quad (83)$$

without generating any terms other than τ^5.

B. Application to system identification of continuous-time models

Since $2n$ parameters $(\alpha_1, \cdots, \alpha_n, \beta_1, \cdots, \beta_n)$ of the sampled-data system $H(z)$ are determined by $n + m + 1$ parameters $\boldsymbol{\pi} = (a_1, \cdots, a_n, b_0, \cdots, b_m)$ of the continuous-time system $G(s)$, we denote those dependences as $\alpha_i(\boldsymbol{\pi})$ and $\beta_i(\boldsymbol{\pi})$. Theorem 1 and 2 imply that they can be approximated by relatively simple polynomials of $\boldsymbol{\pi}$ for small sample time τ. We apply this property to identification of continuous-time systems from discrete-time input-output data of sampled-data systems.

Let's consider a set of input and output data of the discrete-time system $H(z)$: $\{u(k); k = 0, \cdots, N\}$ and $\{y(k); k = 0, \cdots, N-1\}$. Then the data satisfies the equation

$$y(k+n) + \sum_{i=1}^{n} \alpha_i(\boldsymbol{\pi}) y(k+n-i) - \sum_{i=1}^{n} \beta_i(\boldsymbol{\pi}) u(k+n-i) = 0 \quad (84)$$

We denote $\hat{\boldsymbol{\pi}}$ as estimated values of $\boldsymbol{\pi}$ and consider the squared estimation error.

$$J(\hat{\boldsymbol{\pi}}) = \sum_{k=0}^{n} e^2(k) \quad (85)$$

where $e(k) = y(k+n) + \sum_{i=1}^{n} \alpha_i(\hat{\boldsymbol{\pi}}) y(k+n-i) - \sum_{i=1}^{n} \beta_i(\hat{\boldsymbol{\pi}}) u(k+n-i)$. Our identification problem is to find the minimizer of $J(\hat{\boldsymbol{\pi}})$. From Theorem 1 and 2, the function $J(\boldsymbol{\pi})$ is approximated by a simple polynomial of the parameter $\hat{\boldsymbol{\pi}}$ for a small sample time τ.

We apply the steepest decent method with the golden section linear search to finding the minimizer[3], namely

$$\hat{\boldsymbol{\pi}}_{l+1} = \hat{\boldsymbol{\pi}}_l - \eta_l \begin{bmatrix} \frac{\partial J}{\partial \pi_1} \\ \vdots \\ \frac{\partial J}{\partial \pi_n} \end{bmatrix} \quad (l = 0, 1, \cdots) \quad (86)$$

where

$$\frac{\partial J}{\partial \pi_j} = \sum_{k=0}^{n} 2e(k) \cdot \left(\sum_{i=1}^{n} \frac{\partial \alpha_i(\boldsymbol{\pi})}{\partial \pi_j} y(k+n-i) - \sum_{i=1}^{n} \frac{\partial \beta_i(\boldsymbol{\pi})}{\partial \pi_j} u(k+n-i) \right) \quad (87)$$

and the step size η_l is determined by the linear search for sections generated by the golden ratio.

Example 1. *We consider a continuous-time system*

$$G(s) = \frac{b_0}{s^2 + a_1 s + a_2} \quad (88)$$

and the sampled-data system $H(z)$ with a sample time $\tau = 0.01$. The functions $\alpha_i(\hat{\boldsymbol{\pi}})$ and $\alpha_i(\hat{\boldsymbol{\pi}})$ in the criterion (85) are approximated by

$$\alpha_1 = -2 + \tau a_1 + (-a_1^2 + 2a_2)\tau^2/2 + (-3a_1 a_2 + a_1^3)\tau^3/3! + (4a_1^2 a_2 - a_1^4 - 2a_2^2)\tau^4/4! \quad (89)$$

$$\alpha_2 = 1 - a_1\tau + a_1^2\tau^2/2 - a_1^3\tau^3/3! + a_1^4\tau^4/4! \quad (90)$$

$$\beta_1 = b_0\tau^2/2 - a_1 b_0\tau^3/3! + (a_1^2 - a_2) b_0 \tau^4/4! \quad (91)$$

$$\beta_2 = b_0\tau^2/2 - 2a_1 b_0\tau^3/3! + (3a_1^2 - a_2) b_0 \tau^4/4! \quad (92)$$

We estimate a parameter set $\boldsymbol{\pi} = (a_1, a_2, b_0) = (10, 12, 4)$ in the parameter space $[0, 100] \times [0, 300] \times [0, 20]$ from the response of $H(z)$ for the discrete-time input

$$u(t) = \sin(2\pi t) + 2\sin(3\pi t) \quad (93)$$

where $t = k\tau \in [0, 5]$ $(k = 0, 1, \cdots)$. Table II shows the estimation result by the steepest descent method (86). No initial values $\hat{\boldsymbol{\pi}}_0 \in [0, 100] \times [0, 300] \times [0, 20]$ converging to the local minimums other than the true value $\boldsymbol{\pi} = (10, 12, 4)$ is found as far as the author tried.

initial values	estimated values
$\hat{\boldsymbol{\pi}}_0 = (1, 1, 1)$	$\hat{\boldsymbol{\pi}}_{159} = (10.0001, 11.9994, 3.9999)$
$\hat{\boldsymbol{\pi}}_0 = (20, 20, 20)$	$\hat{\boldsymbol{\pi}}_{75} = (9.9983, 12.0014, 3.9994)$
$\hat{\boldsymbol{\pi}}_0 = (20, 20, 1)$	$\hat{\boldsymbol{\pi}}_{149} = (9.9993, 12.0050, 3.9996)$
$\hat{\boldsymbol{\pi}}_0 = (1, 20, 1)$	$\hat{\boldsymbol{\pi}}_{187} = (9.9990, 12.0070, 3.9993)$
$\hat{\boldsymbol{\pi}}_0 = (1, 1, 20)$	$\hat{\boldsymbol{\pi}}_{180} = (10.0006, 11.9988, 4.0001)$

TABLE II. PARAMETER ESTIMATION FOR $\boldsymbol{\pi} = (10, 12, 4)$ UNDER THE STOP CONDITION $J(\boldsymbol{\pi}) \leq 10^{-12}$

Example 2. *We consider a continuous-time system*

$$G(s) = \frac{K}{s(Ts+1)} \cdot \frac{s-q}{s-p} = \frac{b_0 s + b_1}{s(s^2 + a_1 s + a_2)} \quad (94)$$

and the sampled-data system $H(z)$ with a sample time $\tau = 0.01$. The Taylor expansions of the parameter α_i and β_i $(i = 1, 2, 3)$ of $H(z)$ are presented as (12), (13), $\cdots$, (17). By using the low-order terms of Taylor expansions less than 5-th order, we estimate a parameter set $\boldsymbol{\pi} = (a_1, a_2, b_0, b_1) = (3, 5, 2, 4)$ in the parameter space $[0, 10]^4$ from the response of $H(z)$ for the discrete-time input

$$u(t) = \sin(20\pi t) + 2\cos(500\pi t) \quad (95)$$

where $t = k\tau \in [0, 1]$ $(k = 0, 1, \cdots)$. Table III shows the estimation result by the steepest descent method (86).

initial values	estimated values
$\hat{\boldsymbol{\pi}}_0 = (1, 1, 1, 1)$	$\hat{\boldsymbol{\pi}}_{16032} = (3.0011, 4.8856, 2.0000, 4.0026)$
$\hat{\boldsymbol{\pi}}_0 = (1, 1, 10, 10)$	$\hat{\boldsymbol{\pi}}_{4064} = (3.0193, 4.8910, 2.0000, 4.0387)$
$\hat{\boldsymbol{\pi}}_0 = (1, 10, 10, 10)$	$\hat{\boldsymbol{\pi}}_{1937} = (2.9862, 5.0898, 1.9999, 3.9713)$

TABLE III. PARAMETER ESTIMATION UNDER THE STOP CONDITION $J(\boldsymbol{\pi}) \leq 10^{-14}$

IV. CONCLUSION

In this paper, we proved that Taylor expansion of sampled-data systems with respect to the sample time has a regularity for their subscripts and exponents depending on the exponent of the sample time. It is shown that this regularity helps us to reduce symbolic calculation of Taylor expansion. Since the regularity implies that the relation between the continuous-time and discrete-time systems can be approximated by simple polynomial, we successfully applied it to identification of the continuous-time model from discrete-time data of sampled-data systems based on optimization techniques.

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