On the equivalence between stable inversion for nonminimum phase systems and reciprocal transfer functions defined by the two-sided Laplace transform ⋆

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Abstract

In this paper, the two-sided Laplace transform, a classical but not very common mathematical tool, is revived to express the stable inversion for linear nonminimum phase systems that was recently proposed from the viewpoint of state-space representations. It is demonstrated that those two different expressions for the stable inversion are mathematically equivalent. Simple examples are presented to illustrate the two-sided Laplace transform as a direct and intuitive approach to stable inversion. The two-sided Laplace transform approach is also applied to development of an iterative learning control for nonminimum phase systems that needs neither a precise inversion model nor Fourier-Transform computations but instead requires only measuring the system response with time reversals.

Key words: inverse system; inverse transfer function; feedforward compensating; non-minimum phase systems; flexible arms

1 Introduction

System inversion plays crucial roles in many control applications such as perfect tracking, transient response shaping, disturbance attenuation, and noise cancellation. For example, consider shaping the transient response of a plant $G(s)$. Then, if $1/G(s)$ is stable, one can employ $F(s) = M(s)/G(s)$ as a feedforward controller of $G(s)$, where $M(s)$ is a model that has the desired response. If $G(s)$ is a non-minimum phase system or equivalently, $1/G(s)$ has unstable poles, the feedforward controller $F(s) = M(s)/G(s)$ cannot be used as a prefilter. However, substituting the minimizer $F(s) \in RH_\infty$ of $\|G(s)F(s) - M(s)\|_\infty$ for the unstable feedforward controller $F(s) = M(s)/G(s)$ has been proposed. This approach has the advantage that both feedforward and feedback controllers can be designed within the same framework of H_∞ optimization [9]. It is recognized that feedback controller design based on the transfer function is very effective from the viewpoint of robustness or sensitivity. However, effectiveness of the feedforward controller or the prefilter designed by the aforementioned approach is controversial from the viewpoint of response shaping.

On the other hand, it has been shown that the recently developed inversion technique called stable inversion achieves perfect tracking even if $M(s)/G(s)$ has unstable poles; equivalently, $M(s)/G(s) \notin RH_\infty$ [3,20,6,18,4]. This technique utilizes non-causal or preview information about desired trajectories to generate bounded input profiles; therefore the inversion cannot be expressed by transfer functions defined via the conventional one-sided Laplace transform, which was originally employed to analyze feedback control systems where any element is essentially on-line or causal. In most of the work reported upon so far, the stable inversion is expressed by state-space representations with a boundary value condition. This means that stable-inversion-based feedforward controllers lack transfer-function expressions that are widely used by control engineers.

In order to improve this situation, this paper introduces the two-sided Laplace transform to express stable inversion and its calculation by means of the decomposition of the reciprocal transfer function $1/G(s)$. It will be shown that the conventional stable inversion expressed by state-space representations with boundary conditions is mathematically equivalent to the proposed transfer-function expression, which is a more intuitive and direct approach to stable inversion problems, as illustrated by

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examples. It will also be demonstrated that the two-sided-Laplace-transform expression clarifies that non-causal convolution for stable inversion can be replaced by causal convolution with time reversals; this useful property is applied to develop an iterative learning control that needs neither an inversion model nor frequency-domain computations. An experimental application to the tip control of a flexible arm will be presented.

In this paper, we use the following notations: For $f(t) : (-\infty, +\infty) \to R$, norms of $f(t)$ are defined as $\|f(t)\|_\infty := \text{ess sup}_{t\in(-\infty,+\infty)} |f(t)|$, $\|f(t)\|_1 := \int_{-\infty}^{+\infty} |f(t)|dt$ and $\|f(t)\|_2 := \int_{-\infty}^{+\infty} |f(t)|^2 dt$. Spaces of functions which have bounded $\|\cdot\|_1$, $\|\cdot\|_2$ and $\|\cdot\|_\infty$ norms are denoted by L_1, L_2 and L_∞, respectively. For $F(s) : C \to C$, norms of $F(s)$ are defined as $\|F(s)\|_\infty = \text{ess sup}_{\omega\in(-\infty,+\infty)} |F(j\omega)|$ and $\|F(s)\|_2 := \int_{-\infty}^{+\infty} |F(j\omega)|^2 d\omega$.

2 Stable inversion for LTI systems and the two-sided Laplace transform

Consider a plant

$$G(s) = \frac{c_0 s^m + \cdots + c_{m-1}s + c_m}{s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n} \tag{1}$$

with the state-space representation

$$\dot{x} = Ax + Bu \tag{2}$$
$$y = Cx \tag{3}$$

We assume that all poles of $G(s)$ or eigenvalues of A are in the left half plane. Then the reciprocal of the transfer function (1) is

$$\begin{aligned} 1/G(s) = &\left(s^{n-m} + \bar{a}_1 s^{n-m-1} + \cdots + \bar{a}_{n-m}\right)/c_0 \\ &+ \frac{\bar{a}_{n-m+1}s^{m-1} + \cdots + \bar{a}_n}{c_0 s^m + c_1 s^{m-1} + \cdots + c_m} \end{aligned} \tag{4}$$

which has the state-space representation

$$\dot{\xi} = \Phi\xi + \Lambda\eta \tag{5}$$
$$u = \Gamma\xi + \Delta\eta \tag{6}$$

with state vector $\xi = (x_{n-m+1}, \cdots, x_n)^{\mathrm{T}}$ and input vector $\eta = (y^{(n-m)}, \cdots, y^{(1)}, y)^{\mathrm{T}}$ where $y^{(k)} = d^k y/dt^k$ and $\Delta = \left[\, 1/c_0 \;\; \bar{a}_1/c_0 \;\; \cdots \;\; \bar{a}_{n-m}/c_0 \,\right]$; (Φ, Λ, Γ) are the matrices of the controllable canonical form of the last term of (4).

If $G(s)$ is a non-minimum phase system, Φ has eigenvalues in the right half plane and the solution of the initial value problem of (5) is unbounded. This implies that the input to achieve perfect tracking to an output trajectory y_d is unbounded in the causal framework. However, if the constraint of causality is not imposed on (5), there exists a bounded input to achieve perfect tracking for a class of output trajectories.

Proposition 1 *[3,6] Assume that $y_d^{(i)} \in L_1 \cap L_\infty$ ($i = 0, 1, \cdots, n-m$) and that $G(s)$ has no zero on the imaginary axis. Then, there exist bounded $x_d(t)$ and $u_d(t)$ such that*

$$\dot{x}_d = Ax_d + Bu_d \tag{7}$$
$$y_d = Cx_d \tag{8}$$

and

$$u_d(t) \to 0, x_d(t) \to 0 \quad \text{as } t \to \pm\infty \tag{9}$$

□

Remark 2 *It has been shown that the L_1 assumption for $y_d^{(i)}$ can be eliminated [5]. In this paper, however, we discuss stable inversion as defined in Proposition 1 to simplify the Fourier-Laplace analysis for the stable inversion.*

Functions $x_d(t)$ and $u_d(t)$ given in Proposition 1 are obtained by solving the differential equation (5) under the boundary condition (9). The solution is equivalent to one that can be obtained by a change of state coordinates in (5) and a convolution calculation as follows[3]: choose state coordinates $L\bar{\xi} = \xi$ such that

$$\bar{\Phi} = L^{-1}\Phi L = \begin{bmatrix} \bar{\Phi}_- & 0 \\ 0 & \bar{\Phi}_+ \end{bmatrix} \tag{10}$$

where the eigenvalues of $\bar{\Phi}_-$ and $\bar{\Phi}_+$ are in the left and right half planes, respectively. Then, the solution is given by

$$x_d(t) = \int_{-\infty}^{+\infty} \phi(t-\tau)\bar{\Lambda} \left[\, y^{(n-m)}(\tau) \;\; \cdots \;\; y^{(1)}(\tau) \;\; y(\tau) \,\right]^{\mathrm{T}} d\tau \tag{11}$$

where

$$\phi(t) = \begin{bmatrix} 1(t)\exp(\bar{\Phi}_- t) & 0 \\ & -1(-t)\exp(\bar{\Phi}_+ t) \end{bmatrix} \tag{12}$$

and $1(t)$ is the unit step function.

Since the definition of stable inversion published so far is only the one defined by the state space representation of Proposition 1, it is necessary to convert the transfer function (1) into its state space representation whenever stable inversion for linear time-invariant systems is discussed; it is inconvenient to deal with the stable inversion when the plant to be considered is given in terms of the transfer function, which is directly related to frequency domain analysis and consequently is often used by control engineers. In this paper, we show that the reciprocal transfer function (4) in itself actually represents the stable inversion defined in Proposition 1 if we consider the transfer function as being defined by the two-sided Laplace transform[2,14], which, although it is a classical mathematical tool, has not been very widely used in control engineering [1]. The viewpoint based on the two-sided Laplace transform enriches approaches for dealing with stable inversion, as is shown later.

For a function $g(t)$ defined on the infinite time horizon, the two-sided Laplace transform is defined as

$$\mathcal{L}[g(t)](s) = G(s) := \int_{-\infty}^{+\infty} e^{-st} g(t) dt \qquad (13)$$

where the region of convergence is the strip $\{s; \gamma_1 < \mathrm{Re}(s) < \gamma_2\}$ [2,14]. It should be noted that $\mathcal{L}[g'(t)](s) = sG(s)$, $\mathcal{L}\left[\int_{-\infty}^{t} g(\tau)d\tau\right](s) = G(s)/s$ and that

$$\mathcal{L}\left[\int_{-\infty}^{+\infty} g(t-\tau)f(\tau)d\tau\right](s) = G(s)F(s) \qquad (14)$$

where $F(s) = \mathcal{L}[f(t)](s)$. The inverse transform can be expressed as

$$\mathcal{L}^{-1}[G(s)](t) = g(t) = \frac{1}{2\pi j}\int_{\alpha - j\infty}^{\alpha + j\infty} e^{st} G(s) ds$$
$$= \begin{cases} \sum_{\mathrm{Re}(p_n)<\alpha} \mathrm{Res}(e^{st}G(s), p_n) & t \geq 0 \\ \sum_{\mathrm{Re}(p_m)>\alpha} \mathrm{Res}(-e^{st}G(s), p_m) & t < 0 \end{cases} \qquad (15)$$

where $\mathrm{Res}(H(s), p)$ denotes the residue of $H(s)$ at p and $\{p_n\}$ and $\{p_m\}$ are the sets of poles of $G(s)$ that are in the left and right half planes of the vertical line $s = \alpha$ ($\alpha \in (\gamma_1, \gamma_2)$), respectively.

The next theorem shows that the reciprocal of the transfer function based on the two-sided Laplace transform is equivalent to stable inversion as defined in Proposition 1.

Theorem 3 *Assume that* $y_d^{(i)} \in L_1 \cap L_\infty$ $(i = 0, 1, \cdots, n-m)$ *and* $G(s)$ *has no zero on the imaginary axis. Functions* u_d *and* x_d *satisfying (7), (8) and (9) can be defined by*

$$u_d(t) = \mathcal{L}^{-1}[1/G(s) \cdot Y_d(s)] \qquad (16)$$
$$x_d(t) = \mathcal{L}^{-1}[(sI - A)^{-1} B U_d(s)] \qquad (17)$$

where $Y_d(s) = \mathcal{L}[y_d]$ *and* $U_d(s) = \mathcal{L}[u_d]$. □

[1] As far as the author knows, an application to identification of systems with delay was reported in the field of control engineering[7]

PROOF. From the assumption on y_d, we have $\mathcal{L}^{-1}[(s^{n-m} + \bar{a}_1 s^{n-m-1} + \cdots + \bar{a}_{n-m})Y_d(s)] \in L_1 \cap L_\infty$ Since $G(s)$ has no zero on the imaginary axis, the value α of (15) for the fractional term of (4) can be chosen as 0. This implies that $\mathcal{L}^{-1}\left[\frac{\bar{a}_{n-m+1}s^{m-1} + \cdots + \bar{a}_n}{c_0 s^m + c_1 s^{m-1} + \cdots + c_m}\right] \in L_1 \cap L_\infty$; moreover, we also have that $\mathcal{L}^{-1}\left[\frac{\bar{a}_{n-m+1}s^{m-1} + \cdots + \bar{a}_n}{c_0 s^m + c_1 s^{m-1} + \cdots + c_m} Y_d(s)\right] \in L_1 \cap L_\infty$ From these properties, u_d defined by (16) satisfies $u_d(t) \in L_1 \cap L_\infty$. Since all eigenvalues of A are in the left half plane, $u_d(t) \in L_1 \cap L_\infty$ implies that $x_d(t) \in L_1 \cap L_\infty$, which leads to (9). From the definition of $G(s)$ and (A, B, C), u_d and x_d defined by (16) and (17) satisfy (7) and (8). This completes the proof. □

The next corollary shows that stable inversion can be obtained by cascaded convolutions that correspond to the stable and antistable components of the transfer function.

Corollary 4 *Assume the same conditions on* $y_d^{(i)}$ *and* $G(s)$ *as for Theorem 3. Let the last term in (4) be expressed as*

$$\frac{\bar{a}_{n-m+1}s^{m-1} + \cdots + \bar{a}_n}{c_0 s^m + c_1 s^{m-1} + \cdots + c_m} = F_l(s) \cdot F_r(s) \qquad (18)$$

where $F_l(s)$ *and* $F_r(s)$ *are proper transfer functions that have all poles in the left and right half planes, respectively and let* $f_l(t)$ *and* $f_r(t)$ *be the bounded functions that transformed into* $F_l(s)$ *and* $F_r(-s)$ *by the standard one-sided Laplace transform, respectively, i.e.* $F_l(s) = \int_0^{+\infty} e^{-st} f_l(t)dt$ *and* $F_r(-s) = \int_0^{+\infty} e^{-st} f_r(t)dt$.

Then the bounded $u_d(t)$ *defined by (16) can be expressed as*

$$u_d(t) = (y_d^{n-m}(t) + \bar{a}_1 y_d^{n-m-1}(t) + \cdots + \bar{a}_{n-m} y_d(t))/c_0 + \int_{-\infty}^{t} f_l(t-\tau)v(\tau)d\tau \qquad (19)$$

$$v(t) = \left.\int_{-\infty}^{\sigma} f_r(\sigma - \tau) y_d(-\tau) d\tau\right|_{\sigma = -t} \qquad (20)$$

□

PROOF. From (15) with $\alpha = 0$, we have

$$\mathcal{L}^{-1}[F_l(s)] = \begin{cases} \sum_n \mathrm{Res}(e^{st}F_l(s), p_n) & \text{if } t \geq 0 \\ 0 & \text{if } t < 0 \end{cases}$$
$$= \bar{\mathcal{L}}^{-1}[F_l(s)] = f_l(t) \qquad (21)$$

$$\mathcal{L}^{-1}[F_r(-s)] = \begin{cases} \sum_m \mathrm{Res}(e^{st}F_r(-s), -p_m) & \text{if } t \geq 0 \\ 0 & \text{if } t < 0 \end{cases}$$
$$= \bar{\mathcal{L}}^{-1}[F_r(-s)] = f_r(t) \qquad (22)$$

and

$$\mathcal{L}^{-1}[F_r(s)] = -\frac{1}{2\pi j}\int_{+j\infty}^{-j\infty} e^{-st}F_r(-s)ds \quad (23)$$

$$= \begin{cases} 0 & \text{if } t \geq 0 \\ -\sum_m \mathrm{Res}(-e^{-st}F_r(-s), -p_m) & \text{if } t < 0 \end{cases}$$

$$= f_r(-t) \quad (24)$$

These equalities, together with (14) imply (19) and that $v(t) = \int_t^{+\infty} f_r(-(t-\tau))y_d(\tau)d\tau$, which leads to (20). □

Remark 5 *It is straightforward to show that the stable inversion can also be obtained by the decomposition:* $1/G(s) = \bar{F}_l(s)\cdot\bar{F}_r(s)$ *or* $1/G(s) = \bar{H}_l(s) + \bar{H}_r(s)$ *and integration similar to that in Corollary 4.*

Example 6 *Consider a non-minimum phase system*

$$G(s) = \frac{(s+4)(3-s)}{s^3+2s^2+3s+4} \quad (25)$$

and let us obtain the stable inversion for a desired output trajectory $y_d(t)$*. Firstly, we express the reciprocal of (25) as* $1/G(s) = -s-1+F_l(s)\cdot F_r(s)$ *where*

$$F_l(s) = \frac{14s+16}{s+4} = -\frac{50}{s+4}+14,\quad F_r(-s) = \frac{1}{s+3} \quad (26)$$

From (19) and (20), we have the bounded input:

$$u_d(t) = -y_d'(t) - y_d(t) + \int_{-\infty}^{t}(-50e^{-4(t-\tau)})v(\tau)d\tau + 14v(t) \quad (27)$$

$$v(t) = \left.\int_{-\infty}^{\sigma} e^{-3(\sigma-\tau)}y_d(-\tau)d\tau\right|_{\sigma=-t} \quad (28)$$

It should be noted that the function (28) corresponds to the reversed-time signal of the bounded response of $1/(s+3)$ *for the reversed-time trajectory* y_d*; the convolution in (28) is integrated simply in the one direction on the time horizon with the preceding and following time reversals. For numerical computation of* $u_d(t)$*, it is not necessary to calculate the closed forms of the inverse Laplace transforms such as* e^{-3t} *for* $F_r(-s)$ *etc. It is easier to compute the convolutions numerically by the state-space representation for* $F_l(s)$ *and* $F_r(-s)$ *with time reversals, namely*

$$\dot{x}_1(t) = -4x_1(t) + v(t) \quad (29)$$

$$u_d(t) = -50x_1(t) + 14v(t) - y_d'(t) - y_d(t) \quad (30)$$

and

$$\dot{x}_2(t) = -3x_2(t) + y_d(-t) \quad (31)$$

$$v(-t) = x_2(t) \quad (32)$$

for (27) and (28), respectively, the numerical integrations of which can be computed by currently standard software for control engineering, e.g. MATLAB.

In order to compare the proposed method with the standard approach, we apply the state-space representation (5) and (6) with the state coordinate change for (10) and (11): we have $\Delta = \begin{bmatrix} -1 & -1 \end{bmatrix}$ *and the controllable canonical form* (Φ, Λ, Γ) *for* $(14s+16)/(-s^2-s+12)$*, which are transformed into*

$$\dot{\bar{\xi}} = \begin{bmatrix} \bar{\Phi}_- & 0 \\ 0 & \bar{\Phi}_+ \end{bmatrix}\bar{\xi} + \begin{bmatrix} -40/7 & 0 \\ -58/7 & 0 \end{bmatrix}\begin{bmatrix} y \\ y' \end{bmatrix} \quad (33)$$

$$u = \begin{bmatrix} 1 & 1 \end{bmatrix}\bar{\xi} + \begin{bmatrix} -1 & -1 \end{bmatrix}\begin{bmatrix} y \\ y' \end{bmatrix} \quad (34)$$

where $\bar{\Phi}_- = -4$ *and* $\bar{\Phi}_+ = 3$*. Clearly, these expressions correspond to the partial fraction decomposition, namely*

$$1/G(s) = -s - 1 + \frac{-40/7}{s+4} + \frac{-58/7}{s-3} \quad (35)$$

which leads to the same convolution as (11) with (12), namely

$$u_d(t) = -y_d'(t) - y_d(t) + \int_{-\infty}^{t}(-40/7\times e^{-4(t-\tau)})y_d(\tau)d\tau + \left.\int_{-\infty}^{\sigma} 58/7\times e^{-3(\sigma-\tau)}y_d(-\tau)d\tau\right|_{\sigma=-t} \quad (36)$$

As seen above, the known state-space approach to stable inversion with the state coordinate change of (10) is equivalent to the partial fraction decomposition of the reciprocal transfer function. However, the handy approach given in Corollary 4, namely cascaded decomposition into stable and antistable parts, is nontrivial when stable inversion is discussed only from the state space viewpoint of Proposition 1 or the common one-sided Laplace transform viewpoint; Theorem 3 with Corollary 4 yields a more intuitive and direct method to solve the stable inversion problem than conventional approaches [2].

3 An application of the two-sided-Laplace-transform approach to stable inversion

To apply the stable inversion defined in Proposition 1 to practical systems implies the necessity to identify a

[2] e.g. the approach to calculate the closed form for the stable inversion presented in [16], which was originally proposed as the expansion of the optimal output shaping problem[15] to nonminimum-phase systems.

precise model of the objective systems. It is difficult to identify higher-order system models such as mechanical elements with flexible structures. In order to circumvent such difficulties for inversion-based feedforward control, iterative learning control was developed; this is a technique to obtain the input profile that achieves the desired output trajectory through iterative operations on the objective systems[8,10,13,11]. Tien et. al.[19] proposed an inversion-based iterative learning control applied successfully to positioning a flexible AFM piezoscanner, which was analyzed in the frequency domain and implemented with a discrete Fourier transform. Our two-sided-Laplace-transform approach developed in the previous section reveals an iterative learning control that needs neither the inversion model nor Fourier-Transform computations, but which requires only measurement of the system response with time reversals.

Theorem 7 *Let $y_d(t)$ be a desired output trajectory and assume that there exists $u_d(t) = \mathcal{L}^{-1}[1/G(s) \cdot Y_d(s)] \in L_1 \cap L_\infty$ where $Y_d(s) = \mathcal{L}[y_d(t)]$ and $G(s)$ has no zeros on the imaginary axis and $G(j\omega) \to 0$ as $|\omega| \to \infty$. Then, the iterative learning control defined by*

$$U_{k+1}(s) = U_k(s) - \alpha G(-s)\{G(s)U_k(s) - Y_d(s)\} \quad (37)$$

with $U_0(s) \equiv 0$ generates the input sequence $\{U_k(s); k = 0, 1, \cdots\}$ satisfying

$$\|U_k(s) - 1/G(s) \cdot Y_d(s)\|_2 \to 0 \quad \text{as } k \to \infty \quad (38)$$

where α is a constant satisfying $0 < \alpha < 1/\|G(s)\|_\infty^2$.

PROOF. Let $E_k(s) = U_k(s) - 1/G(s) \cdot Y_d(s)$. Then we have $E_{k+1}(s) = (1 - \alpha G(-s)G(s))E_k(s)$, which leads to $E_{k+1}(j\omega) = (1 - \alpha|G(j\omega)|^2)E_k(j\omega)$. Moreover $|E_k(j\omega)|^2 = (1 - \alpha|G(j\omega)|^2)^{2k}|E_0(j\omega)|^2$. Since $1 - \alpha|G(j\omega)|^2 < 1$ for $\omega \in (-\infty, +\infty)$, we have $\|E_k(s)\|_2^2 \leq \int_{-\Omega}^{+\Omega}(1 - \alpha|G(j\omega)|^2)^{2k}|E_0(j\omega)|^2 d\omega + \int_{\Omega<|\omega|\leq\infty}|E_0(j\omega)|^2 d\omega$ for any positive Ω. Note that $\mathcal{L}^{-1}[1/G(s) \cdot Y_d(s)] \in L_1 \cap L_\infty$ implies that $E_0(s) = -1/G(s) \cdot Y_d(s) \in L_2$. Then we have $\int_{\Omega<|\omega|\leq\infty}|E_0(j\omega)|^2 d\omega \to 0$ as $\Omega \to \infty$. Hence, for any given $\epsilon > 0$, there exist Ω and a positive integer N such that $\|E_k(s)\|_2^2 < \epsilon/2 + \epsilon/2$, for all $k > N$. This completes the proof. □

It should be noted that the time-domain counterpart for $G(-s)$ is the time-reversed signal of the response of $G(s)$ for the time-reversed input signal, which is shown by the same discussion as for Corollary 4 and Example 6. The iterative algorithm described by the transfer function in Theorem 7 can be replaced with the following simple time-domain algorithm.

Corollary 8 *Assume the same properties as for Theorem 7 and that all poles of $G(s)$ are in the left half plane. The iterative learning control defined by*

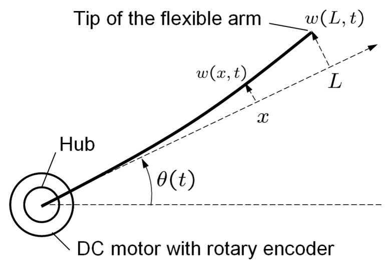

Fig. 1. Experimental setup of the flexible arm

$$\eta_k(t) = \int_{-\infty}^{t} g(t-\tau)u_k(\tau)d\tau - y_d(t) \quad (39)$$

$$u_{k+1}(t) = u_k(t) - \alpha \int_{-\infty}^{\sigma} g(\sigma-\tau)\eta_k(-\tau)d\tau \Big|_{\sigma=-t} \quad (40)$$

generates the input sequence $\{u_k(t); k = 0, 1, \cdots\}$ that satisfies $\|u_k(t) - u_d(t)\|_2 \to 0$ as $k \to \infty$ where $u_d(t) = \mathcal{L}^{-1}[1/G(s) \cdot Y_d(s)]$.

PROOF. The assumption of poles of $G(s)$ with (14) and (15) implies (39). Since all poles of $G(-s)$ are in the right half plane, (40) is obtained by the same discussion as in the proof of Corollary 4. By the Parseval equality, the norm (38) can be replaced with the norm in the time domain. □

The iterative algorithm presented above is implemented by the following procedures:

(1) measure the error signal η_k between the system response for the input u_k and y_d, i.e. (39)
(2) measure the system response for the time-reversed error signal η_k, i.e. the convolution in (40) and record the time-reversed signal of the measured response
(3) update the input as in (40) and go back to the first step

Example 9 *We apply the iterative learning control given by Corollary 8 to tip control of the experimental flexible arm depicted in Fig. 1 (The length of the arm: $L = 3.0 \times 10^{-1}$m, the moment of inertia of the arm including the hub: $I = 637.4 \times 10^{-6}$Kg $\cdot$ m^2). The hub is directly driven by a DC motor; the rotational angle $\theta(t)$ with respect to the inertial reference frame is measured by a rotary encoder embedded in the motor. The deflection of the tip $w(L, t)$ with respect to the frame fixed on the hub is measured by an optical device on the hub, which senses the horizontal location of a light source attached to the tip. Assuming that the deflection of the tip is sufficiently smaller than the length of the*

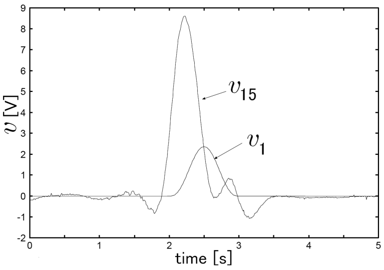

Fig. 2. Input profiles for $k = 1$ and $k = 15$

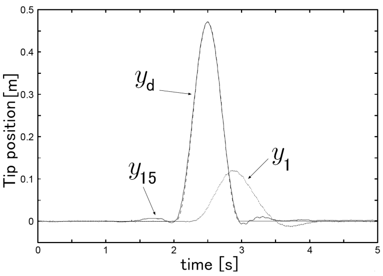

Fig. 3. Output for $k = 1$ and $k = 15$ with the desired trajectory

arm, we consider $y(t) = L\theta(t) + w(L, t)[m]$ *as the position of the tip. The dynamics of the motor is expressed by* $T(s) = K_m/R \cdot V_{in}(s) - K_m^2/R \cdot \dot{\theta}(s)$ *where* $K_m = 7.67 \times 10^{-3}\text{Nm/A}$ *and* $R = 2.60\Omega$*. Observations based on the Euler-Bernoulli model lead to transfer functions of the flexible arm as follows[1]:* $\dot{\theta}(s)/T(s) = 1/(Is) + 1/I \sum_{i=1}^{\infty}(a_i s)/(s^2 + 2\zeta_i\omega_i s + \omega_i^2)$ *and* $y(s)/T(s) = P(s) = L/(Is^2) + 1/I \sum_{i=1}^{\infty} k_i/(s^2 + 2\zeta_i\omega_i s + \omega_i^2)$*. A PD-feedback* $V_{in}(t) = -K_P\theta(t) - K_D\dot{\theta}(t) + v(t)$ *is applied to the aforementioned experimental setup where* $v(t)$ *is the reference input. The PD gain* K_P *and* K_D *are experimentally chosen to make the system stable. Letting a desired trajectory of the tip position be* $y_d(t) = \pi L/2\{-f(t)^6 + 3f(t)^4 - 3f(t)^2 + 1\}$ *for* $t \in [2, 3]$ *and* 0 *for* $t \in [0, 5]\backslash[2, 3]$ *where* $f(t) = 2(t - 2.5)$*, an experiment was conducted to apply the iterative learning control to update the reference input* $v(t)$ *with respect to the trajectory of the error* $y(t) - y_d(t)$ *without identifying the transfer function* $G(s) = \mathcal{L}[y(t)]/\mathcal{L}[v(t)]$*. The length of the time horizon for the experiment was chosen sufficiently long in view of the interval of the support of* $y_d(t)$ *and the decay time of the impulse response of* $G(s)$ *estimated by experiment. Figures 2 and 3 show the input* v_k *and tip position* y_k *with* y_d*, respectively, for the number of iterations* $k = 1$ *and* $k = 15$*. Fig. 4 shows the tip deflection* $w(L, t)$ *with the tip position* y_k *for* $k = 15$*[12].*

4 Conclusion

In this paper, the two-sided Laplace transform that is a classical but not very common mathematical tool was revived to express the stable inversion for linear non-minimum phase systems that has been recently proposed

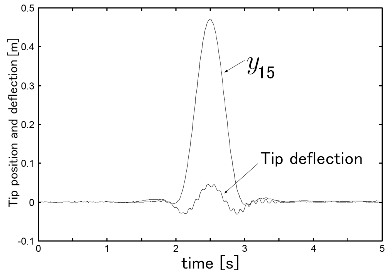

Fig. 4. Output with the tip deflection for $k = 15$

from the viewpoint of state-space representations. It was shown that those two different expressions for stable inversion are mathematically equivalent. Simple examples were presented to illustrate the two-sided Laplace transform as a direct and intuitive approach to stable inversion. The two-sided Laplace transform approach was also applied to development of an iterative learning control for non-minimum phase systems that needs neither a precise inversion model nor Fourier-Transform computations but which requires only measurement of the system response with time reversals.

It is straightforward to apply the idea proposed in this paper to discrete-time or sampled-data systems by introducing the two-sided z-transform. This helps us to clarify the relation between inversion of sampled-data and continuous-time systems, the former of which have mostly unstable zeros, even if the latter have no unstable zeros. In another paper, the author proved that the former with a small sample time approximates the latter[17].

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