Approach to relocate sampled zeros for feedforward control application

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***Abstract*—This paper presents the Taylor expansion of all the zeros of sampled-data systems with respect to the sample time τ. Using this expansion, we have developed a method to relocate sampled zeros to facilitate feedforward control based on pole-zero cancellation. The proposed approach has been successfully applied to the feedforward control of a DC motor by using an analog compensator for zero relocation. Illustrative experiments have also been described.**

I. Introduction

In recent times, control systems for various applications have usually been implemented as digital control systems; such systems are continuous-time systems having samplers and zero-order holds of the same time period and are essentially hybrids of continuous- and discrete-time systems. A conventional approach to the analysis and design of such systems is based on theories of discrete-time systems that describe the behavior on the sample times. Although the theories for linear discrete-time systems are mostly compatible with those for linear continuous-time systems, some of their critical theoretical relationships are still ambiguous. One such relationship is the correspondence of the zeros of transfer functions; the location of the sampled zero with respect to the sample time is given by an intricate function. This hinders the application of inversion-based feedforward control to digital control systems. To illustrate this problem, let us consider the following example:

Example 1: The continuous-time model of a simple DC motor is given by

$$G(s) = \frac{1}{s(s+1)} \tag{1}$$

Using the zero-order hold and a sampler having sample time $\tau = 0.1$, a sampled-data system

$$H(z) = \frac{4.8374 \times 10^{-3}(z + 0.9672)}{(z-1)(z-0.9048)} \tag{2}$$

is obtained; this system has the sampled zero at -0.9048, which is very close to -1. Let

$$M(z) = \frac{0.17z}{z^2 - 1.32z + 0.5} \tag{3}$$

represent a model that has the desired response and consider feedforward control based on pole-zero cancellation for the sampled-data system $H(z)$, namely, $F(z) = M(z)H(z)^{-1}$

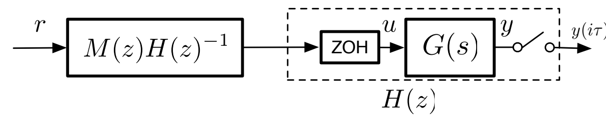

Fig. 1. Feedforward control for sampled-data systems

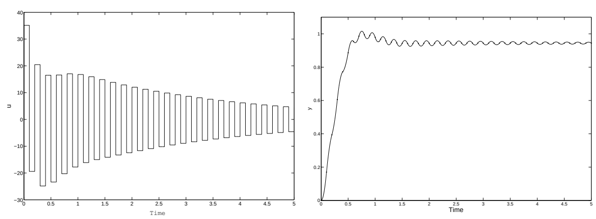

Fig. 2. The step response of $M(z)H(z)^{-1}$ and the output of $G(s)$

(see Fig. 1). Then, a typical response of this feedforward controller is a large oscillation having long decay time (Fig. 2, left); this leads to inter-sample ripples in the system response (Fig. 2, right).

It is known that for various control applications the transfer function of sampled-data systems usually have unstable zeros. This makes it difficult to apply feedforward control to such systems. A possible approach to overcome this difficulty is the relocation of the sampled zeros by adjusting the parameters of the system. In fact, a general continuous-time system

$$G(s) = \frac{K(s-q_1)\cdots(s-q_m)}{(s-p_1)(s-p_2)\cdots(s-p_n)} \tag{4}$$

leads to the sampled-data system

$$H_\tau(z) = \frac{C_\tau\{z-\gamma_1(\tau)\}\cdots\{z-\gamma_{n-1}(\tau)\}}{(z-e^{p_1\tau})(z-e^{p_2\tau})\cdots(z-e^{p_n\tau})} \tag{5}$$

the poles and zeros of which are functions of the parameters of $G(s)$. In contrast to the poles, however, there is no simple relation between the m continuous-time zeros $\{q_1, \cdots, q_m\}$ and the $n-1$ sampled zeros $\{\gamma_1(\tau), \cdots, \gamma_{n-1}(\tau)\}$, which are generally not expressed as closed formulae. Although the Taylor expansion of the sampled zeros is partially given by

$$\gamma_k(\tau) = 1 + q_k\tau + \frac{q_k^2\tau^2}{2} + O(\tau^3) \tag{6}$$

TABLE I
ZEROS OF THE EULER-FROBENIUS POLYNOMIAL

$n-m-1$	zeros
1	-1
2	$-2-\sqrt{3}, (-2-\sqrt{3})^{-1}$
3	$-5-2\sqrt{6}, -1, (-5-2\sqrt{6})^{-1}$
$\vdots$	$\vdots$
odd	$\lambda_{i1}, \cdots, \lambda_{i(i-2)/2}, -1, \lambda_{i(i-2)/2}^{-1}, \cdots, \lambda_{i1}^{-1}$ $(\lambda_{i1} < \cdots < \lambda_{i(i-2)/2} < -1)$
even	$\lambda_{i1}, \cdots, \lambda_{i(i-1)/2}, \lambda_{i(i-1)/2}^{-1}, \cdots, \lambda_{i1}^{-1}$ $(\lambda_{i1} < \cdots < \lambda_{i(i-1)/2} < -1)$

for $k = 1, \cdots, m$ [1], the expansion of the other $n-m-1$ sampled zeros has not yet been derived. This hinders the development of techniques to relocate the sampled zeros. To the best of the authors' knowledge, no research has been conducted in this regard.

However, recent research has shown that the limit of the sampled zeros, as $\tau \to 0$, exhibits a regular property[2], [3]. On the basis of this property, we can derive the Taylor expansion formulae of the zeros with respect to the sample time and apply the formulae to relocate the zeros approximately to facilitate feedforward control. The remainder of this paper is organized as follows. In Section 2, we derive the Taylor expansion of the zeros. In Section 3, we apply the expansion to the relocation of the zeros for a DC motor. Finally, in Section 4, we conclude the paper.

II. Taylor expansion of all sampled zeros

It is known that the limit of the sampled zeros has the following property:

Proposition 1: [2], [3] As $\tau \to 0$,

$$H_\tau(z) \to \frac{\tau^{n-m}(z-1)^m B_{n-m}(z)}{(z-1)^n} \tag{7}$$

where $B_{n-m}(z)$ is the Euler-Frobenius polynomial of the degree $n-m-1$, given by

$$B_{n-m}(z) = b_1^{n-m} z^{n-m-1} + b_2^{n-m} z^{n-m-2} + \cdots + b_{n-m}^{n-m} \tag{8}$$

where

$$b_k^{n-m} = \sum_{l=1}^{k} (-1)^{k-l} l^{n-m} \begin{pmatrix} n-m+1 \\ n-m-l \end{pmatrix}, \quad k = 1, \cdots, n-m. \tag{9}$$

□

It is known that the zeros of the Euler-Frobenius polynomial (Table I) exhibit the following regularities[4]:

1) Every zero is a zero of order 1 on the negative real axis.
2) If λ is a zero, then $1/\lambda$ is also a zero.

From these properties, we derive the Taylor expansion of the sampled zeros.

Theorem 1: Let the transfer function of the sampled-data system (5) be expressed as

$$H_\tau(z) = \frac{\tau^{n-m}\{F_0(z) + \tau F_1(z) + \tau^2 F_2(z) + \cdots\}}{(z - e^{p_1\tau}) \cdots (z - e^{p_n\tau})} \tag{10}$$

Then, the sampled zero that tends to the zero of the Euler-Frobenius polynomial, λ, is expanded as

$$\gamma(\tau) = \lambda + \alpha\tau + \beta\tau^2 + O(\tau^3) \tag{11}$$

where

$$\alpha = -\frac{F_1(\lambda)}{F_0'(\lambda)} \tag{12}$$

$$\beta = -\frac{\alpha^2 F_0''(\lambda) + 2\alpha F_1'(\lambda) + 2F_2(\lambda)}{2F_0'(\lambda)} \tag{13}$$

Proof: Letting $\tau = 0$ for the first and second derivatives of

$$\begin{aligned} &F_0\left(\lambda + \alpha\tau + \beta\tau^2 + O(\tau^3)\right) \\ &+ \tau F_1\left(\lambda + \alpha\tau + \beta\tau^2 + O(\tau^3)\right) \\ &+ \tau^2 F_2\left(\lambda + \alpha\tau + \beta\tau^2 + O(\tau^3)\right) + O(\tau^3) = 0 \end{aligned} \tag{14}$$

leads to $\alpha F_0'(\lambda) + F_1(\lambda) = 0$ and $2\beta F_0'(\lambda) + \alpha^2 F_0''(\lambda) + 2\alpha F_1'(\lambda) + 2F_2(\lambda) = 0$, respectively, which reduce to (12) and (13) if $F_0'(\lambda) \neq 0$.

From Proposition 1, we have $F_0(z) = K(z-1)^m B_{n-m}(z)/(n-m)!$. As every zero of the Euler-Frobenius polynomial $B_{n-m}(z)$ is a zero of order 1 on the negative real axis[4], we have $F_0'(\lambda) \neq 0$. ■

The polynomials $F_0(z)$, $F_1(z)$, and $F_2(z)$ in Theorem 1, i.e., the Taylor expansions of the numerator of $H_\tau(z)$, are calculated as described in Lemma 1, which can be easily programmed using a symbolic computation software, e.g., Maple or Matlab symbolic math toolbox.

Lemma 1: Assume that $G(s)/s$ has no multiple pole and let $r_l = G(s) \cdot (s - p_l)/s|_{s=p_l}$ $(l = 0, 1, \cdots)$, where $p_0 = 0$ and $\{p_1, \cdots, p_n\}$ are the poles of $G(s)$. Then

$$F_k(z) = \frac{1}{(n-m+k)!} \sum_{j=1}^{n} c(k,j)(-1)^j z^{n-j} \tag{15}$$

$$c(k,j) = \sum_{l=0}^{n} r_l \sum_{\substack{\text{for all combinations of j numbers} \\ \{i_1, \cdots, i_j\} \text{ chosen from} \\ \{0, 1, \cdots, n\} \setminus \{l\}}} (p_{i_1} + \cdots + p_{i_j})^{n-m+k} \tag{16}$$

□

Remark 1: If $G(s)$ has a double pole at p, we can still apply Lemma 1 to $G(s)$ approximately by substituting slightly different poles p and $p+\epsilon$ for the double pole. The assumption in Lemma 1 does not prevent us from applying the result to major applications such as the one described in Example 1.

III. Application to relocation of sampled zeros

For the case of the relative degree $n-m = 2$, the expansion (11) of the sampled zero can be simplified as

$$\gamma(\tau) = -1 + \kappa\tau - \frac{1}{2}\kappa^2\tau^2 + O(\tau^3) \tag{17}$$

where $\kappa = \{(q_1 + \cdots + q_m) - (p_1 + \cdots + p_n)\}/3$ [1]. On the basis of (17), another element $C(s)$ with free parameters q_0 and p_0 is added to the objective system $G(s)$ as

$$C(s)G(s) = \frac{s - q_0}{s - p_0} \times \frac{K(s - q_1)\cdots(s - q_{n-2})}{(s - p_1)\cdots(s - p_n)} \tag{18}$$

the sampled zeros of which can be easily relocated by adjusting the values of q_0 and p_0. The zeros of the sampled-data system derived from (18) are given by

$$\begin{aligned}\gamma(\tau) =& -1 + \left(\kappa + \frac{q_0 - p_0}{3}\right)\tau - \frac{1}{2}\left(\kappa + \frac{q_0 - p_0}{3}\right)^2 \tau^2 \\ &+ O(\tau^3) \end{aligned} \tag{19}$$

$$\gamma_k(\tau) = 1 + q_k\tau + \frac{q_k^2\tau^2}{2} + O(\tau^3) \quad (k = 0, 1, \cdots, n-2) \tag{20}$$

Neglecting $O(\tau^3)$, we can adjust the locations of $\gamma(\tau)$ and $\gamma_0(\tau)$ by setting the values of q_0 and p_0. In order to maximize the convergence rate of the poles added to cancel the zeros $\gamma(\tau)$ and $\gamma_0(\tau)$, we minimize the values of $|\gamma(\tau)|$ and $|\gamma_0(\tau)|$ to $1/2$, which can be achieved by setting

$$(q_0, p_0) = \left(-\frac{1}{\tau}, -\frac{4}{\tau} + 3\kappa\right) \tag{21}$$

Example 2: Consider a DC motor with an inertial load:

$$G(s) = \frac{1.35 \times 10^5}{s(s + 5.3)} \tag{22}$$

where the input and output are input voltage [V] and the pulse count of a rotary encoder [count] (2048 counts = 2π rad), respectively. When $\tau = 0.01$, the sampled-data system is given by

$$H(z) = \frac{6.6323(z + 0.9825)}{(z - 1)(z - 0.9484)} \tag{23}$$

The sampled zero at -0.9825 is not desirable from the viewpoint of the convergence rate required for feedforward control. Hence, before the DC motor, we insert an operational amplifier circuit corresponding to $C(s)$ described above (see Fig. 3).

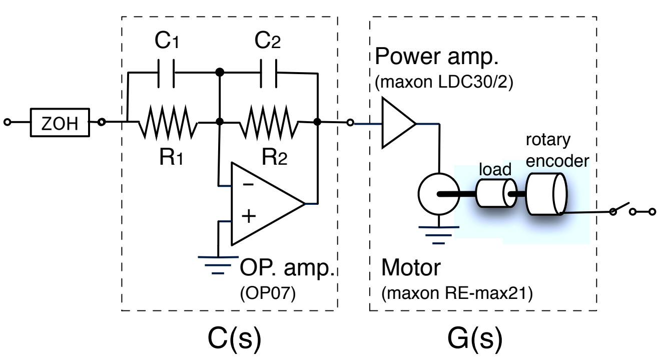

Fig. 3. Motor with power and operational amplifiers

[1]The authors validated this result for $n \leq 6$ by using symbolic computation software.

As $\tau = 0.01$ and $\kappa = \{-0 - (-5.3)\}/3$ for the DC motor and

$$C(s) = -\frac{C_1}{C_2} \times \frac{s + 1/(C_1R_1)}{s + 1/(C_2R_2)} \tag{24}$$

for the circuit, the optimum values of (q_0, p_0) are

$$(q_0, p_0) = (-100, -394.7) \tag{25}$$

which are realized by setting $C_1 = C_2 = 0.1$ μF, $R_1 = 100$ kΩ, $R_2 = 25.4$ kΩ for the capacitors and resistors. Applying the compensator $C(s)$, we have the sampled-data system

$$\bar{H}(z) = \frac{3.5364(z + 0.4493)(z - 0.368)}{(z - 1)(z - 0.9484)(z - 0.01861)} \tag{26}$$

which is more suitable for the pole-zero cancellation of feedforward control than the one given by (23).

To demonstrate the effectiveness of the zero-relocation technique described above, experiments of feedforward control with and without $C(s)$ were conducted (see Fig. 4). It should be noted that $1/z$ is added to the feedforward controller to let the transfer function to be a proper one. Figs. 5 and 6 show the input and output signals, respectively, of the DC motor with the zero compensator $C(s)$ when a step function was applied as the reference or desired response r_i. Figs. 7 and 8 show those without the zero compensator. Although there exists a steady-state error in the former case, which can be suppressed by applying feedback compensation, the shape of the response is similar to that of the reference (Fig. 6). On the other hand, there is significant oscillation in the input signal to the DC motor in the latter case (Fig. 7). This oscillation excites the unmodeled dynamics of the DC motor, thereby resulting a large deviation of the output from the reference (Fig. 8).

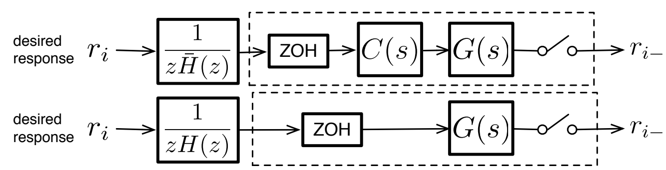

Fig. 4. Feedforward control for the DC motor with and without compensation

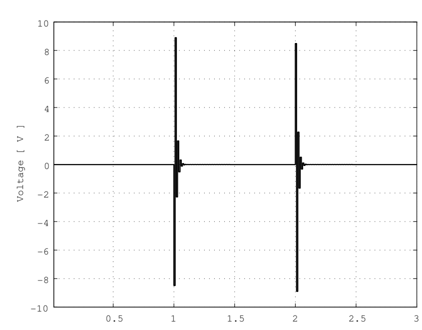

Fig. 5. Input signal for the step reference of the compensated motor

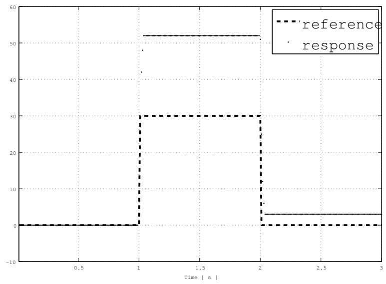

Fig. 6. Response for the step reference of the compensated motor

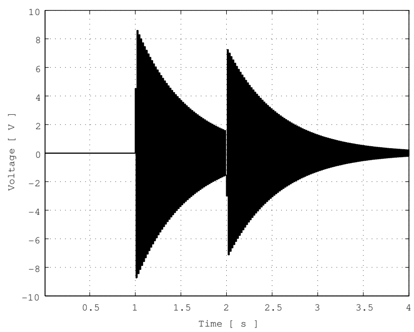

Fig. 7. Input signal for the step reference of the uncompensated motor

IV. Application to design of multi-rate sampled-data systems

Fig. 9 shows the inter-sample output signals of the zero compensator $C(s)$, which reveals that $C(s)$ plays the role of a signal filter in the step function of zero order hold (ZOH). This indicates that a sufficiently fast ZOH can be substituted for $C(s)$ in order to approximate the shape of the inter-sample signal. A ZOH having sample time τ and $C(s)$ are replaced by a fast ZOH having the sample time $\delta = \tau/n$ and n discrete-time systems $\{J_0(z), \cdots, J_{n-1}(z)\}$, where n is an integer(see Fig. 10).

V. Conclusion

This paper presented the Taylor expansion of all the sampled zeros with respect to the sample time τ. On the basis of this expansion, we proposed a technique to relocate the sampled zeros approximately for continuous-time systems with relative degree $n - m = 2$. Moreover, we proposed the use of a compensator to relocate the sampled zeros for DC motor applications; the compensator is realized as an operational amplifier circuit. The results of our experiments were presented to illustrate the effectiveness of the compensator for the relocation of the sampled zeros. In the future work, we plan to design multi-rate sampled-data systems on the basis of the approach described in this paper.

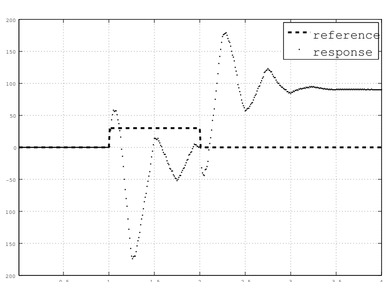

Fig. 8. Response for the step reference of the uncompensated motor

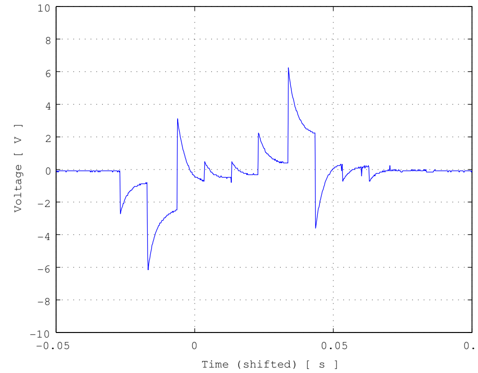

Fig. 9. Output signal from the compensator $C(s) = \frac{s-q}{s-p}$

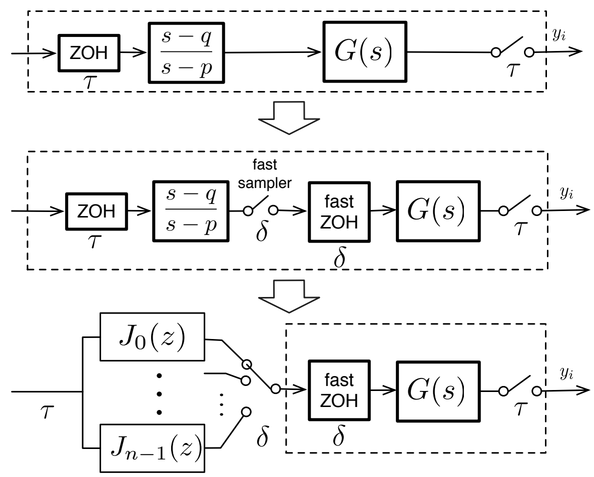

Fig. 10. Substitution of the compensator with a fast-rate sampled-data system

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