A regularity of Taylor expansion of sampled zeros and its application to adaptive control

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Abstract: This paper shows a regularity of Taylor expansion of sampled zeros which implies that the approximated value of the sampled zeros is dominated by the coefficients of the highest-order terms in the numerator and denominator of the continuous-time system. Moreover, it is shown that the first-order term of any discretization zero is expressed by a linear formula of the summations of all poles and zeos. Numerical examples are presented to demonstrate successful applications to stabilizing sampled zeros.

Keywords: sampled zero, sampled-data systems, digital control

1. INTRODUCTION

Control systems for various applications have usually been implemented as digital control system; such systems are continuous-time systems having samplers and zero-order holds. A basic approach to the analysis and design of such systems is based on theories for linear discrete-time systems that describe the behavior on the sample times. Zeros of those discrete-time systems are called sampled zeros. It is well-known that sampled zeros are not simply related with the zeros of the original continuous-time systems. To make matters worse, sampled zeros are often unstable even if the continuous-time zeros are stable.

Example 1: Consider a DC motor:

$$G(s) = \frac{1}{s(s+1)} \tag{1}$$

Using the zero-order hold and a sampler having sample time $\tau = 0.1$, we have a discrete-time system:

$$H(z) = \frac{4.837 \times 10^{-3}(z+0.967)}{(z-1)(z-0.905)} \tag{2}$$

If control approaches based on pole-zero cancellation such as feedforward control or adaptive control are applied to $H(z)$, the unstable phenomenon called ringing occurs because the zero at -0.965 is very close to -1.■

Unfortunately, there is no general closed formula expressing sampled zeros. However, it is known that Taylor expansion of sampled zeros with respect to the sample time is simpler than expected.

Example 2: Consider a continuous-time system

$$G(s) = \frac{s+b_1}{s^3+a_1s^2+a_2s+a_3} \tag{3}$$
$$= \frac{s+b_1}{(s-p_1)(s-p_2)(s-p_3)} \tag{4}$$

† Takuya Sogo is the presenter of this paper.

Then we have the discrete-time system

$$H(z) = \frac{\beta_1(\tau)z^2+\beta_2(\tau)z+\beta_3(\tau)}{z^3+\alpha_1(\tau)z^2+\alpha_2(\tau)z+\alpha_3(\tau)} \tag{5}$$
$$= \frac{\beta_1(\tau)(z-\gamma_1(\tau))(z-\gamma_2(\tau))}{(z-e^{p_1\tau})(z-e^{p_2\tau})(z-e^{p_3\tau})} \tag{6}$$

where τ is the sample time. While the poles are expressed as the closed formula, the sampled zeros $\gamma_1(\tau)$ and $\gamma_2(\tau)$ cannot be simply expressed. However, we have Taylor expansion of the sampled zeros as

$$\gamma_1(\tau) = 1 - b_1\tau + b_1^2\tau^2 - b_1^3\tau^3 + \cdots \tag{7}$$

$$\begin{aligned}\gamma_2(\tau) =& -1 + (a_1-b_1)\tau/3 \\ &+ (2a_1b_1 - b_1^2 - a_1^2)\tau^2/9 \\ &+ (5b_1^3 + 6a_2b_1 - 6a_1^2b_1 + 3a_1a_2 + a_1^3 \\ &- 9a_3)\tau^3/45 + \cdots\end{aligned} \tag{8}$$

which have a simple regularity on the order of τ and the subscripts, i.e. the sum of the subscript of a_i and b_i in the term of τ^k is equal to the order k: e.g. only a_1 or b_1 emerges in the term of τ^1; the coefficients for τ^2 consist only of $b_1 \times b_1$, $a_1 \times b_1$ or $a_1 \times a_1$, the sum of whose subscripts is equal to 2; the sum of the subscripts of the coefficient for τ^3 is restricted to be 3.■

Moreover, one can demonstrate that such regularity holds true for systems with higher degrees in the numerator or denominator. One can expect that the approximated value of the sampled zero is generally dominated by the value of the coefficient a_i or b_i with the smaller number of the subscript i.

In this paper, we show the above-mentioned regularity on sampled zeros for general systems. Moreover, we discuss applications to adjusting the sampled zeros for stable pole-zero cancellation.

2. TAYLOR EXPANSION OF SAMPLED ZEROS

We consider SISO continuous-time system

$$G(s) = \frac{b_0 s^m + b_1 s^{m-1} + \cdots + b_m}{s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n} \tag{9}$$

or

$$\begin{aligned} \dot{\boldsymbol{x}}(t) &= A_c \boldsymbol{x}(t) + b_c u \\ y &= c\boldsymbol{x}(t) \end{aligned} \tag{10}$$

where $G(s) = c(sI - A_c)^{-1} b_c$. In the following discussions, we assume $b_0 = 1$ for simplicity without loss of generality for study on sampled zeros. The continuous-time system (10) connected to the zero-order hold and a sampler having sample time τ leads to discrete-time system

$$\begin{aligned} \boldsymbol{\xi}(k+1) &= A\boldsymbol{\xi}(k) + bv(k) \\ w(k) &= c\boldsymbol{\xi}(k) \end{aligned} \tag{11}$$

where

$$\begin{bmatrix} A & b \\ \mathbf{0} & 1 \end{bmatrix} = \exp\left(\begin{bmatrix} A_c & b_c \\ \mathbf{0} & 0 \end{bmatrix} \tau \right) \tag{12}$$

The transfer function is expressed by

$$H(z) = \frac{\beta_1 z^{n-1} + \beta_2 z^{n-2} + \cdots + \beta_n}{z^n + \alpha_1 z^{n-1} + \cdots + \alpha_{n-1} z + \alpha_n} \tag{13}$$

where

$$H(z) = c(zI - A)^{-1} b = \frac{\begin{vmatrix} A - zI & b \\ c & 0 \end{vmatrix}}{\left| A - zI \right|} \tag{14}$$

The formula (12) leads to Taylor expansion of the discrete-time coefficient β_i $(i = 1, 2, \cdots, n)$ which is expressed by the continuous-time coefficients b_i $(i = 1, \cdots, m)$ or a_i $(i = 1, \cdots, n)$.

Example 3: The coefficients in the numerator (5) are expressed as follows:

$$\begin{aligned} \beta_1 =& \tau^2/2 - (a_1 - b_1)\tau^3/3! \\ &+ \left(a_1{}^2 - a_1 b_1 - a_2\right)\tau^4/4! \\ &- \left(a_1{}^3 - a_1{}^2 b_1 - 2\,a_1 a_2 \right. \\ &\left. + a_2 b_1 + a_3\right)\tau^5/5! + \cdots \end{aligned} \tag{15}$$

$$\begin{aligned} \beta_2 =& -(a_1 - 4\,b_1)\tau^3/3! + \left(2a_1^2 - 8\,a_1 b_1\right)\tau^4/4! \\ &- \left(3\,a_1{}^3 - 13\,a_1{}^2 b_1 - a_1 a_2 \right. \\ &\left. + 8\,a_2 b_1 - 2\,a_3\right)\tau^5/5! + \cdots \end{aligned} \tag{16}$$

$$\begin{aligned} \beta_3 =& -\tau^2/2 + (2\,a_1 + b_1)\tau^3/3! \\ &- \left(3\,a_1{}^2 + 3\,a_1 b_1 - a_2\right)\tau^4/4! \\ &+ \left(4\,a_1{}^3 + 6\,a_1{}^2 b_1 - 3\,a_1 a_2 \right. \\ &\left. - a_2 b_1 - a_3\right)\tau^5/5! + \cdots \end{aligned} \tag{17}$$

■

It should be noted that there is no term of τ^0 or τ^1 in the above example.

It was shown that for any n and m there is has no term lower than τ^{n-m} . Hence, the numerator of $H(z)$ is expressed as $F_0(z)\tau^{n-m} + F_1(z)\tau^{n-m+1} + F_2(z)\tau^{n-m+2} + \cdots$ or

$$H(z) = \frac{\tau^{n-m}\Phi(z)}{z^n + \alpha_1 z^{n-1} + \cdots + \alpha_{n-1} z + \alpha_n} \tag{18}$$

where

$$\Phi(z) = F_0(z) + F_1(z)\tau^1 + F_2(z)\tau^2 + \cdots \tag{19}$$

and $F_i(z)$ $(i = 0, 1, \cdots)$ is $n-1$-degree polynomial of z independent from τ.

Example 4: From the results of Example 3, the numerator (5) is expressed as $\tau^2\Phi(z)$ where

$$F_0(z) = (z^2 - 1)/2 \tag{20}$$

$$\begin{aligned} F_1(z) =& \left\{-(a_1 - b_1)z^2 - (a_1 - 4b_1)z \right. \\ &\left. + (2a_1 + b_1)\right\}/3! \end{aligned} \tag{21}$$

$$\begin{aligned} F_2(z) =& \left\{(a_1^2 - a_1 b_1 - a_2)z^2 + (2a_1^2 - 8a_1 b_1)z \right. \\ &\left. + (3a_1^2 + 3a_1^2 b_1 - a_2)\right\}/4! \end{aligned} \tag{22}$$

■

Let Taylor expansion of the sampled zero $\gamma(\tau)$ be

$$\gamma(\tau) = \lambda + \Lambda_1\tau + \Lambda_2\frac{\tau^2}{2!} + \cdots + \Lambda_k\frac{\tau^k}{k!} + \cdots \tag{23}$$

Then Λ_k is expressed as

$$\Lambda_k = \left.\frac{d^k}{d\tau^k}\gamma(\tau)\right|_{\tau=0} \tag{24}$$

which can be obtained from $\left.\frac{d^k}{d\tau^k}\Phi(\gamma(\tau))\right|_{\tau=0}$ because we have $\Phi(\gamma(\tau)) \equiv 0$ by the definition and $\frac{d^k}{d\tau^k}\Phi(\gamma(\tau)) \equiv 0$ for any positive integer k. From the equation $\left.\frac{d^k}{d\tau^k}\Phi(\gamma(\tau))\right|_{\tau=0} = 0$ where $k = 1, 2$ and 3, we have

$$\Lambda_1 = -F_1(\lambda)/F_0^{(1)}(\lambda) \tag{25}$$

$$\begin{aligned} \Lambda_2 =& -\left\{\Lambda_1 F_0^{(2)}(\lambda) + 2\Lambda_1 F_1^{(1)}(\lambda) + F_2(\lambda) \right. \\ &\left. \right\}/F_0^{(1)}(\lambda) \end{aligned} \tag{26}$$

$$\begin{aligned} \Lambda_3 =& -\left\{3\Lambda_1\Lambda_2 F_0^{(2)}(\lambda) + \Lambda_1^3 F_0^{(3)}(\lambda) \right. \\ &+ 2\Lambda_2 F_1^{(1)}(\lambda) + (3\Lambda_1^2 + \Lambda_2)F_1^{(2)}(\lambda) \\ &\left. + 6\Lambda_1 F_2^{(1)}(\lambda) + 6F_3(\lambda)\right\}/F_0^{(1)}(\lambda) \end{aligned} \tag{27}$$

It is noted that $F_0^{(1)}(\lambda) \neq 0$ because λ is the zero of $F_0(z)$ and defined by (20). Hence, the expressions (25), (26) and (27) with $\lambda = 1$ or -1 lead to the expansions (7) or (8), respectively.

The general term Λ_k is recursively expressed by $\Lambda_1, \cdots, \Lambda_{k-1}$ and $F_i^{(j)}$ $(i = 1, \cdots, k,\ j = 0, 1, \cdots, k)$, namely

$$\Lambda_k = -\left[\sum_{(\boldsymbol{m_{k-1}})} \frac{k!}{c(\boldsymbol{m_{k-1}})} F_0^{(m_1+\cdots+m_{k-1})}(\lambda) \prod_{j=1}^{k-1} \Lambda_j^{m_j} + k! F_k(\lambda) \right.$$

$$+ \sum_{l=1}^{k-1} \left\{ {}_kC_l \left(\sum_{(\boldsymbol{m_{k-l}})} \frac{(k-l)!}{c(\boldsymbol{m_{k-l}})} F_l^{(m_1+\cdots+m_{k-l})}(\lambda) \right.\right.$$

$$\left.\left.\left. \times \prod_{j=1}^{k-l} \Lambda_j^{m_j} \right) l! \right\} \right] \left\{ F_0^{(1)}(\lambda) \right\}^{-1} \qquad (28)$$

where $\sum_{(\boldsymbol{m_k})}$ indicates the summation for all nonnegative k-tuples $(m_1, \cdots, m_k)$ satisfying

$$1 \cdot m_1 + 2 \cdot m_2 + \cdots + k \cdot m_k = k \qquad (29)$$

and the positive integer $c(\boldsymbol{m_k})$ is defined by $c(\boldsymbol{m_k}) = m_1!1!^{m_1} \cdots m_k!k!^{m_k}$.

3. REGULARITY IN THE EXPANSION OF SAMPLED ZERO

From the definition (12), (13) and (14), the coefficient β_i $(i = 1, \cdots, n)$ is expressed by

$$\beta_i = \sum_{k=0}^{\infty} \tau^k \sum_{j=0}^{m} f_{i,k,j}(a_1, \cdots, a_n) b_j \qquad (30)$$

where $f_{i,k,1}, \cdots, f_{i,k,m}$ are polynomials of $a_1, \cdots, a_n$, i.e.

$$f_{i,k,j}(a_1, \cdots, a_n) = \sum d_{i,k,j}(l_1, \cdots, l_n) a_1^{l_1} \cdot \cdots \cdot a_n^{l_n} \qquad (31)$$

where $d_{i,k,j}(l_1, \cdots, l_n)$ is rational constant and the sum is over all n-tuples of nonnegative integers $(l_1, \cdots, l_n)$ satisfying the inequality

$$l_1 + l_2 + \cdots + l_n \leq k \qquad (32)$$

As is seen in Example 3, the sum is actually restricted to the much smaller set of the n-tuples than the inequality (32). It was shown that such restriction holds true for any n and m.

Theorem 1: [3] The sum (31) is restricted to n-tuples of nonnegative integers $(l_1, \cdots, l_n)$ satisfying

$$1 \cdot l_1 + 2 \cdot l_2 + \cdots + n \cdot l_n = k - (n - m) - j \qquad (33)$$

■

Example 5: Consider the same example as Example 2 or 3. The number of 3-tuples (l_1, l_2, l_3) satisfying the inequality (32) is 1, 3, 6, 15$(= 3 + 6 + 3 + 3)$ and 21$(= 3+6+6+3+3)$ for $k = 0, 1, 2, 3, 4$ and 5, respectively; the number of the combination explodes as k increases.

k	$b_0 (= 1)$ $(j = 0)$ l_1	l_2	l_3
0	-	-	-
1	-	-	-
2	0	0	0
3	1	0	0
4	0	1	0
	2	0	0
5	0	0	1
	1	1	0
	3	0	0

k	b_1 $(j = 1)$ l_1	l_2	l_3
0	-	-	-
1	-	-	-
2	-	-	-
3	0	0	0
4	1	0	0
5	0	1	0
	2	0	0

Table 1 Nonnegative integers (l_1, l_2, l_3) satisfying (33) $(n = 3, m = 1)$

k	l_1	l_2	l_3	ν_1
1	1	0	0	0
	0	0	0	1
2	2	0	0	0
	0	1	0	0
	1	0	0	1
	0	0	0	2
3	3	0	0	0
	1	1	0	0
	0	1	0	1
	2	0	0	1
	1	0	0	2
	0	0	0	3

k	l_1	l_2	l_3	ν_1
4	4	0	0	0
	2	1	0	0
	0	2	0	0
	1	0	1	0
	3	0	0	1
	1	1	0	1
	0	0	1	1
	2	0	0	2
	0	1	0	2
	1	0	0	3
	0	0	0	4

Table 2 Nonnegative integers (l_1, l_2, l_3, ν_1) satisfying the equality (33) $(n = 3, m = 1)$

However, the result in Theorem 1 suppresses the increase of the combination very much as is seen in Table 1. ■

Theorem 2: Let $\gamma(\tau)$ be a sample zero that goes to λ which is a zero of the polynomial $F_0(z)$ as $\tau \to 0$. Then assuming λ is not a multiple zero of the polynomial $F_0(z)$, Taylor expansion of the sample zero $\gamma(\tau)$ is expressed by

$$\gamma(\tau) = \lambda + \sum_{k=1}^{\infty} \left\{ \tau^k \sum c_k(l_1, \cdots, l_n, \nu_1, \cdots, \nu_m) a_1^{l_1} \cdots a_n^{l_n} b_1^{\nu_1} \cdots b_m^{\nu_m} \right\} \qquad (34)$$

where the sun is over all $n + m$-tuples of nonnegative integers $(l_1, \cdots, l_n)$ and $(\nu_1, \cdots, \nu_m)$ satisfying

$$1 \cdot l_1 + \cdots + n \cdot l_n + 1 \cdot \nu_1 + \cdots + m \cdot \nu_m = k \qquad (35)$$

and $c_k(l_1, \cdots, l_n, \nu_1, \cdots, \nu_m)$ is real constant. ■

Example 6: Consider the same example as Example 2 or 3. $(3 + 1)$-tuples (l_1, l_2, l_3, ν_1) satisfying the equality (35) are listed in Table 2. Theorem 2 implies that the sampled zero should be expressed by

$$\gamma(\tau) = \lambda + \tau (c_{11} a_1 + c_{12} b_1) + \tau^2 \left(c_{21} a_1^2 + c_{22} a_1 b_1 + c_{23} a_2 + c_{24} b_1^2 \right) + O(\tau^3) \qquad (36)$$

where $\lambda = -1$ or 1 and c_{ij} $(i = 1, 2,\ j = 1, 2, 3, 4)$ are real constants. Those expressions mean that the value of the sampled zero is dominated by the coefficient a_1 and b_1. It should be noted that the above-mentioned property is consistent with the concrete expression (7) or (8) calculated by the formula (25) and (26).■

It should be noted that the polynomial $F_0(z)$ defining λ in Theorem 2 is generally expressed by

$$F_0(z) = (z-1)^m B_{n-m-1}(z)/(n-m)! \qquad (37)$$

where $B_i(z)$ $(i = 0, 1, \cdots)$ is i-degree Euler-Frobenius polynomial, all zeros of which are known to be single and negative real[1][2]. Hence the assumption in Theorem 2 is satisfied for any sampled zero $\gamma(\tau)$ that tends to the zero of the Euler-Frobenius polynomial $B_i(z)$ which is called discretization zero.

Corollary 1: For any $n \geq m+2 \geq 2$, the discretization zero is expressed

$$\begin{aligned}\gamma(\tau) =& \lambda + \tau\left(c_{11}a_1 + c_{12}b_1\right) + \tau^2\left(c_{21}a_1^2 + c_{22}a_1b_1\right.\\ &\left. + c_{23}a_2 + c_{24}b_1^2 + c_{25}b_2\right) + O(\tau^3) \qquad (38)\end{aligned}$$

where c_{ij} $(i = 1, 2,\ j = 1, \cdots, 5)$ are real constants; the term that includes b_i $(i = 1$ or $2)$ is eliminated if $m = 0$ or 1 and b_i is undefined. ■

4. APPLICATION TO STABILIZING ZEROS

Consider model following control (MFC) depicted in Fig.1 for a discrete-time system $H(z) = B(z)/A(z)$ where

$$B(z) = \beta_1 z^{n-1} + \beta_2 z^{n-2} + \cdots + \beta_n \qquad (39)$$

$$A(z) = z^n + \alpha_1 z^{n-1} + \cdots + \alpha_{n-1}z + \alpha_n \qquad (40)$$

The MFC makes the output y follow the output of the desired model $H_m(z)$ where the polynomial $D(z) = z^n + d_1 z^{n-1} + \cdots + d_n$ represents the characteristic polynomial of the closed loop. Moreover, one can consider

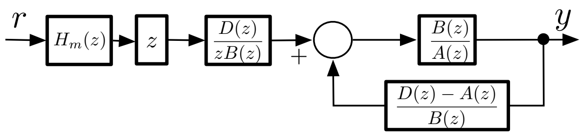

Fig. 1 Model following control

extension of the MFC to adaptive MFC based on RLS estimation for $B(z)$ and $A(z)$. Since the MFC or adaptive MFC relies on pole-zero cancellation, stability of zeros is indispensable for application.

Example 7: Consider applying MFC or adaptive MFC to the DC motor model (1) in Example 1. Since the discrete-time model (2) has the zero at -0.965, the pole that cancels the zero causes so-called ringing phenomena which should be avoided from the stability viewpoint. To do so, we consider a filter

$$C(s) = \frac{s-q}{s-p} \qquad (41)$$

with free negative parameters q and p to be inserted between the zero-order hold and $G(s)$, namely

$$C(s)G(s) = \frac{s-q}{s(s+1)(s-p)} \qquad (42)$$

We can estimate the zeros for (42) by the Taylor expansion (7) and (8) by letting

$$(b_1, a_1, a_2, a_3) = (-q, 1-p, -p, 0) \qquad (43)$$

From the expression of the terms of the first order, we expect that $|\gamma_1| < 1$ and $|\gamma_2| < 1$ when we choose (q, p) satisfying $q < 0$ and $q > p - 1$. By letting $(q, p) = (-5, -10)$ or $(-20, -40)$, we have

$$H_{CG}(z) = \frac{0.0041937(z+0.8229)(z-0.6066)}{(z-1)(z-0.9048)(z-0.3679)} \qquad (44)$$

or

$$H_{CG}(z) = \frac{0.0033241(z+0.6404)(z-0.1434)}{(z-1)(z-0.9048)(z-0.01832)} \qquad (45)$$

whose sampled zeros can be canceled by the stable poles.■

It should be noted that the above example is the application of a special case of Corollary 1. The first-order approximation of any discretization zero is expressed by the linear formula of the coefficient a_1 and b_1, or equivalently the summation of all continuous-time poles p_i $(i = 1, \cdots, n)$ and zeros q_i $(i = 1, \cdots, m)$, respectively, namely

$$\gamma(\tau) = \lambda + \left(c_1 \sum_{i=1}^{n} p_i + c_2 \sum_{i=1}^{m} q_i\right)\tau + O(\tau^2) \qquad (46)$$

where c_1 and c_2 are constants; we expect that the above-mentioned filter $C(s)$ is successfully applied to adjusting general discretization zeros.

5. CONCLUSION

It is shown that the k-th order term of Taylor expansion of sampled zeros is expressed only by the coefficients of the continuous-time system with the subscripts restricted by k. This implies that the first-order approximation term of any discretization zero is determined by a linear function of the summation of all poles and zeros of the continuous-time system.

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