# Taylor series expansion for zeros of sampled-data systems

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***Abstract*—The paper introduces key properties useful for the computation of Taylor expansion of discretization zeros or single intrinsic zeros of sampled-data systems. Furthermore, regularity is shown to exist in the suffixes of the coefficients of the expansion expressions, which implies that the sampling zeros or every coefficient in the numerator of the transfer function of the sampled-data systems is dominated by the coefficients of higher-order terms in the numerator and denominator of the continuous-time counterpart. Moreover, it is shown that the regularity can reduce the calculation effort for expansions of higher-order systems.**

***Index Terms*—Sampling zero, Sampled-data system, Digital control, Inverse system, Multivariate polynomial ring**

## I. Introduction

The linear dynamic system theory has been exhaustively researched in the field for both continuous-time and discrete-time systems. As most controllers for recent industrial applications are implemented as digital computer systems, discretization based on the sample and hold operations is an indispensable part of control systems. Therefore many interesting discrete-time systems are necessarily related to continuous-time systems. However, this relation is not as simplistic. Let us consider a single-input-single-output linear time-invariant system $(A_c, B_c, C)$ with a transfer function

$$G(s) = C(sI - A_c)^{-1}B_c \tag{1}$$

$$= \frac{b_0 s^m + b_1 s^{m-1} + \cdots + b_{m-1}s + b_m}{s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n} \tag{2}$$

$$= \frac{b_0(s-q_1)\cdots(s-q_m)}{(s-p_1)(s-p_2)\cdots(s-p_n)} \tag{3}$$

where $p_1, \cdots, p_n$ are assumed to be all distinct. (it can be assumed that $b_0 = 1$ without loss of generality when studying zeros.) Then, the discrete-time system generated by the sampler and a zero-order hold of sample time $\tau$ is represented by the system matrices $(A, B, C)$, where

$$\begin{bmatrix} A & B \\ \mathbf{0} & 1 \end{bmatrix} = \exp\left(\begin{bmatrix} A_c & B_c \\ \mathbf{0} & 0 \end{bmatrix}\tau\right) = \sum_{k=0}^{\infty}\frac{\tau^k}{k!}\begin{bmatrix} A_c & B_c \\ \mathbf{0} & 0 \end{bmatrix}^k \tag{4}$$

and the transfer function is

$$H(z) = C(zI - A)^{-1}B = \frac{N(z,\tau)}{D(z,\tau)} \tag{5}$$

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where

$$N(z,\tau) = \begin{vmatrix} A - zI & B \\ C & 0 \end{vmatrix} \tag{6}$$

$$= \beta_1(\tau)z^{n-1} + \beta_2(\tau)z^{n-2} + \cdots + \beta_n(\tau) \tag{7}$$

$$D(z,\tau) = |A - zI| \tag{8}$$

$$= z^n + \alpha_1(\tau)z^{n-1} + \cdots + \alpha_{n-1}(\tau)z + \alpha_n(\tau) \tag{9}$$

It is well-known that $H(z)$ is expressed as

$$H(z) = \frac{C_\tau\{z - \gamma_1(\tau)\}\cdots\{z - \gamma_{n-1}(\tau)\}}{(z - \exp(p_1\tau))\cdots(z - \exp(p_n\tau))} \tag{10}$$

[1], [2] (see (52) and (54) in Appendix A) While the poles of the transfer function $H(z)$ are simply denoted as $\{\exp(p_1\tau), \cdots, \exp(p_n\tau)\}$, there is no simple correspondence between the $n-1$ zeros of the transfer function $H(z)$, called sampling zeros, and the $m$ continuous-time counterparts. Although there is no general closed formula for sampling zeros, the limit of the zeros, as the sample time $\tau$ tends to 0, is known to be

$$\lim_{\tau\to 0}\frac{H(z)}{\tau^{n-m}} = \frac{b_0(z-1)^m B_{n-m}(z)}{(n-m)!(z-1)^n} \tag{11}$$

where $B_{n-m}(z)$ represents the Euler-Frobenius or reciprocal polynomials [2], [3], which have a zero at $-1$ when $n-m$ is even and zeros at $\lambda$ and $1/\lambda$, where $\lambda$ is a negative real number less than $-1$. This means that the discrete-time system often has unstable zeros for a small sample time, even if the continuous-time system has no unstable zeros. As the stability of the zeros is of considerable importance for controller design based on pole-zero cancellation, several studies have been conducted to clarify the approximation formula for the zeros. The $m$ intrinsic zeros $\{\gamma_1(\tau), \cdots, \gamma_m(\tau)\}$ that tend to 1 have been shown to be asymptotically evaluated by the exponential functions of the continuous-time counterparts, namely, $\{\exp(q_1\tau), \cdots, \exp(q_m\tau)\}$ [4]. Moreover, a Taylor expansion of the single intrinsic zero with respect to the sample time is presented up to the third-order term [5], [6]. A first-order Taylor expansion of the $n-m-1$ discretization zeros $\{\gamma_{m+1}(\tau), \cdots, \gamma_{n-1}(\tau)\}$ that tend to the zeros of the polynomial $B_{n-m}(z)$ has been presented with a regular property [6]. To the best of the authors' knowledge, no general formula for higher-order Taylor expansion of discretization zeros is known with a regular property. In the present paper, we introduce a general formula for Taylor expansion of discretization zeros with a regular property. For example, we consider the case in which $(n,m) = (3,1)$ and assume $b_0 = 1$. We obtain a Taylor expansion of the discretization zeros $\gamma_2(\tau)$ which will be discussed in Example 2 with 1:

$$\begin{aligned}\gamma_2(\tau) =& -1 + (a_1 - b_1)\tau/3 - (a_1^2 - 2a_1b_1 + b_1^2)\tau^2/9 \\ &+ (a_1^3 - 6a_1^2b_1 + 5b_1^3 + 3a_1a_2 + 6a_2b_1 \\ &- 9a_3)\tau^3/270 + \cdots \end{aligned} \quad (12)$$

As indicated by this example, each sum of the suffixes of the coefficients $a_i$ and $b_j$ existing in the $k$-th-order term $\tau^k$ is equal to $k$, e.g., the coefficient for $\tau^2$ in equation (12) is a linear combination of $a_1a_1$, $a_1b_1$, and $b_1b_1$, the sum of whose suffixes is equal to 2; the coefficient for $\tau^3$ is a linear combination of $a_1a_1a_1$, $a_1a_1b_1$, $a_1a_2$, etc., the sum of the suffixes of which is equal to 3. The present paper shows that such regularity exists for Taylor expansions of any discretization zeros, and therefore, the locations of those zeros are dominated by the value of continuous-time parameters with smaller suffixes. Moreover, for the same example we can obtain a Taylor expansion of the intrinsic zero $\gamma_1(\tau)$, which will be discussed in Example 3 with 1:

$$\begin{aligned}\gamma_1(\tau) =& 1 - b_1\tau + b_1^2\tau^2/2 - b_1^3\tau^3/6 + b_1^4\tau^4/24 \\ &- (10b_1^5 - 5a_1b_1^4 + a_1^2b_1^3 + 4a_2b_1^3 - a_1a_2b_1^2 \\ &- 4a_3b_1^2 + a_1a_3b_1)\tau^5/720 + \cdots \end{aligned} \quad (13)$$

For this expression, the above-mentioned regularity on the suffixes holds true. The present paper also shows that the same regularity exists for Taylor expansion of any single intrinsic zero, i.e. the case where $m = 1$. Finally, it is shown that the regularity can reduce the effort required for the symbolic calculation of the Taylor expansion.

## II. Taylor expansion of zeros

### A. Power series expansion of the numerator

In order to obtain the Taylor expansion of zeros, we prepare a key property of the power series expansion of the numerator $N(z,\tau)$ with respect to the sample time $\tau$, which is denoted by

$$N(z,\tau) = \sum_{k=0}^{\infty} K_k(z)\tau^k \quad (14)$$

The exact expressions $K_k(z)$ for $k = 0, \cdots, k_{\max}$ are symbolically calculated through the following procedure:

*Procedure 1:*

1) Prepare the matrices $(A_c, B_c, C)$ that consist of symbols $b_0, b_1, \cdots, b_m, a_1, \cdots, a_n$, 1, and 0. (e.g. controllable or observable canonical form[1])
2) Symbolically calculate the power series (4) truncated at $k = k_{\max}$ and let them be $(A, B)$.
3) Calculate the matrix determinant (6) and order the expression with respect to $\tau^k$ [2].

[1] Although any realization $(A_c, B_c, C)$ is admissible, a sparse realization such as controllable or observable canonical form is preferable from the viewpoint of computational load for the following steps.

[2] In order to obtain $K_0(z), K_1(z), \cdots, K_{k_{\max}}(z)$ defined by the matrix determinant (6), it suffices to truncate the power series (4) at $k = k_{\max}$ and let them be $(A, B)$ for the determinant (6). It should be noted, however, that the computing of determinant based on such matrices $(A, B)$ may partially generate higher terms than $\tau^{k_{\max}}$. Eliminating such higher terms in the computing process will save memory space.

The above procedure implies that $K_k(z)$ is expressed by

$$K_k(z) = \sum_{i=1}^{n}\sum_{j=0}^{m} f_{i,k,j}(a_1, \cdots, a_n) b_j z^{n-i} \quad (15)$$

where the polynomial $f_{i,k,j}(a_1, \cdots, a_n)$ of $a_1, \cdots, a_n$ is defined by

$$\begin{aligned} &f_{i,k,j}(a_1, \cdots, a_n) \\ &= \sum_{\max(\nu_1+\cdots+\nu_n)=k} c_{i,k,j}(\nu_1, \cdots, \nu_n) \cdot a_1^{\nu_1} \cdot a_2^{\nu_2} \cdot \cdots \cdot a_n^{\nu_n} \end{aligned} \quad (16)$$

where the sum is for $n$-tuples of nonnegative integers $(\nu_1, \cdots, \nu_n)$ satisfying the constraint $\max(\nu_1 + \cdots + \nu_n) = k$, and the coefficients $c_{i,k,j}(\nu_1, \cdots, \nu_n)$ are rational constants. As far as the above-mentioned definitions of the expressions $f_{i,k,j}(a_1, \cdots, a_n)$ or $K_k(z)$ are considered, these exact expressions are expected to become more complicated as the order $k$ or $n$ increases, because the number of combinations of the sum in equation (16) is $\sum_{l=0}^{k}(l + n - 1)!/\{l!(n-1)!\}$. However, the terms that exist in the expressions $f_{i,k,j}(a_1, \cdots, a_n)$ or $K_k(z)$ are actually limited to a fewer combinations:

*Lemma 1:* Summation (16) is limited to the combinations of the $n$-tuples $(\nu_1, \cdots, \nu_n)$ that satisfy

$$1 \cdot \nu_1 + 2 \cdot \nu_2 + \cdots + n \cdot \nu_n = k - (n - m) - j \quad (17)$$

*Proof:* See Appendix A. ■

Lemma 1 implies that there exists no combination of $n$-tuples $(\nu_1, \cdots, \nu_n)$ of nonnegative integers for summation (16) for any nonnegative $j$ when $k < n - m$:

*Corollary 1:* The numerator $N(z,\tau)$ of the transfer function $H(z)$ is represented as

$$N(z,\tau) = \sum_{k=n-m}^{\infty} K_k(z)\tau^k \quad (18)$$

When $k = n - m$, only the $n$-tuple $(\nu_1, \cdots, \nu_n) = (0, \cdots, 0)$ with $j = 0$ forms summation (16) and $f_{i,k,j} \equiv 0$ for $j > 0$. Equivalently, $K_{n-m}(z)$ is a polynomial of $z$ with constant coefficients multiplied by $b_0$. These implications are consistent with equation (11).

*Example 1:* Consider the case where $(n,m) = (3,1)$, which was mentioned in Section I. Then, we estimate the possible combinations for constraint (17) of Lemma 1 as $(\nu_1, \nu_2, \nu_3, j) = (1,0,0,0)$ and $(0,0,0,1)$ for $k = 3$; $(\nu_1, \nu_2, \nu_3, j) = (2,0,0,0)$, $(1,0,0,1)$ and $(0,1,0,0)$ for $k = 4$; $(\nu_1, \nu_2, \nu_3, j) = (3,0,0,0)$, $(2,0,0,1)$, $(1,1,0,0)$, $(0,1,0,1)$ and $(0,0,1,0)$ for $k = 5$. On the other hand, through exact symbolic calculations based on Procedure 1 with $k_{\max} = 5$, we obtain $N(z,\tau) = \beta_1(\tau)z^2 + \beta_2(\tau)z + \beta_3(\tau)$, where

$$\begin{aligned}\beta_1(\tau) =& b_0\tau^2/2 - (a_1 b_0 - b_1)\tau^3/3! \\ &+ \left({a_1}^2 b_0 - a_1 b_1 - a_2 b_0\right)\tau^4/4! \\ &- \left({a_1}^3 b_0 - {a_1}^2 b_1 - 2\,a_1 a_2 b_0 \right.\\ &\left. + a_2 b_1 + a_3 b_0\right)\tau^5/5! + \cdots \end{aligned} \tag{19}$$

$$\begin{aligned}\beta_2(\tau) =& -(a_1 b_0 - 4\,b_1)\tau^3/3! + \left(2a_1^2 b_0 - 8\,a_1 b_1\right)\tau^4/4! \\ &- \left(3\,{a_1}^3 b_0 - 13\,{a_1}^2 b_1 - a_1 a_2 b_0 \right.\\ &\left. + 8\,a_2 b_1 - 2\,a_3 b_0\right)\tau^5/5! + \cdots \end{aligned} \tag{20}$$

$$\begin{aligned}\beta_3(\tau) =& -b_0\tau^2/2 + (2\,a_1 b_0 + b_1)\tau^3/3! \\ &- \left(3\,{a_1}^2 b_0 + 3\,a_1 b_1 - a_2 b_0\right)\tau^4/4! \\ &+ \left(4\,{a_1}^3 b_0 + 6\,{a_1}^2 b_1 - 3\,a_1 a_2 b_0 \right.\\ &\left. - a_2 b_1 - a_3 b_0\right)\tau^5/5! + \cdots \end{aligned} \tag{21}$$

which means we have, for example, $K_0(z) = K_1(z) = 0$,

$$K_2(z) = b_0(z^2-1)/2 = b_0(z-1)B_2(z)/2 \tag{22}$$

$$K_3(z) = \left\{(a_1 b_0 - b_1)z^2 + (a_1 b_0 - 4b_1)z + (2a_1 b_0 + b_1)\right\}/3!. \tag{23}$$

The results of the exact calculations are consistent with the combinations anticipated by Lemma 1.

### *B. Taylor expansion of discretization zeros*

By Corollary 1, we define $\Phi(z,\tau)$ as $\Phi(z,\tau) = N(z,\tau)/\tau^{n-m}$ or

$$\Phi(z,\tau) = K_{n-m}(z) + K_{n-m+1}(z)\tau + K_{n-m+2}(z)\tau^2 + \cdots \tag{24}$$

for notational simplicity. Note that $K_{n-m}(z) = b_0(z-1)^m B_{n-m}(z)/(n-m)!$ according to equation (11). Now, we are ready to examine the Taylor expansion of discretization zeros $\gamma_i(\tau)$ $(i = n-m,\cdots,n-1)$ for the numerator $N(z,\tau)$, which are equivalent to zeros for the function $\Phi(z,\tau)$. The zeros of the Euler-Frobenius polynomials $B_k(z)$ for odd and even $k$ are denoted as $\{\zeta_1,\cdots,\zeta_{(k-1)/2}, 1/\zeta_{(k-1)/2},\cdots,\cdots,1/\zeta_1\}$ and $\{\zeta_1,\cdots,\zeta_{(k-2)/2},-1,1/\zeta_{(k-2)/2},\cdots,\cdots,1/\zeta_1\}$, respectively, where $\zeta_i$ are negative real numbers satisfying $\zeta_i < \zeta_j < -1$ $(i<j)$ [3]. This implies that the discretization zeros are single, negative, and real for a sufficiently small sample time $\tau$. This ensures that the function $\Phi(z,\tau)$ satisfies

$$\left.\frac{\partial}{\partial z}\Phi(z,\tau)\right|_{(z,\tau)=(\lambda,0)} = K^{(1)}_{n-m}(\lambda) \neq 0 \tag{25}$$

where $\lambda$ is the relevant zero of the Euler-Frobenius polynomial $B_{n-m}(z)$, i.e., $\Phi(\lambda,0) = K_{n-m}(\lambda) = 0$. By the implicit function theorem, there exists a class $C^\infty$ function $z = \gamma(\tau)$ for the discretization zero, i.e., $\Phi(\gamma(\tau),\tau) = 0$ for sufficiently small $\tau$. Let us denote the Taylor expansion of the discretization zero $\gamma(\tau)$ with respect to $\tau$ as

$$\gamma(\tau) = \lambda + \sum_{k=1}^{\infty}\Lambda_k\tau^k/k! \tag{26}$$

Then, we can calculate an exact expression of $\Lambda_k = \left.\frac{d^k}{d\tau^k}\gamma(\tau)\right|_{\tau=0}$ from the expression of the polynomial $K_k(z)$ by utilizing the identities $\Phi(\gamma(\tau),\tau) \equiv 0$ and $\left.\frac{\partial^i}{\partial\tau^i}\Phi(\gamma(\tau),\tau)\right|_{\tau=0} = 0$, where $i = 1,2,\cdots$. By $\left.\frac{\partial}{\partial\tau}\Phi(\gamma(\tau),\tau)\right|_{\tau=0} = \left\{\frac{d}{d\tau}\gamma(\tau)\frac{\partial}{\partial z}\Phi(z,\tau) + \frac{\partial}{\partial\tau}\Phi(z,\tau)\right\}_{\tau=0,z=\lambda} = 0$, we have $\Lambda_1 K^{(1)}_{n-m}(\lambda) + K_{n-m+1}(\lambda) = 0$, which yields

$$\Lambda_1 = -K_{n-m+1}(\lambda)/K^{(1)}_{n-m}(\lambda) \tag{27}$$

By $\left.\frac{\partial^2}{\partial\tau^2}\Phi(\gamma(\tau),\tau)\right|_{\tau=0} = 0$, we have $\Lambda_2 K^{(1)}_{n-m}(\lambda) + \Lambda_1^2 K^{(2)}_{n-m}(\lambda) + 2\Lambda_1 K^{(1)}_{n-m+1}(\lambda) + 2K_{n-m+2}(\lambda) = 0$, which yields

$$\begin{aligned}\Lambda_2 = -&\left\{\Lambda_1^2 K^{(2)}_{n-m}(\lambda) + 2\Lambda_1 K^{(1)}_{n-m+1}(\lambda)\right.\\ &\left. + 2K_{n-m+2}(\lambda)\right\}/K^{(1)}_{n-m}(\lambda)\end{aligned} \tag{28}$$

By repeating such calculations, we can recursively obtain the following exact expression for $\Lambda_k$ from $\{\Lambda_1,\cdots,\Lambda_{k-1}\}$ and $\{K_{n-m}(z),\cdots,K_{n-m+k}(z)\}$:

*Lemma 2:* The coefficient $\Lambda_k$ for the Taylor expansion (26) of the discretization zero is calculated by

$$\begin{aligned}\Lambda_k =& -\left\{K^{(1)}_{n-m}(\lambda)\right\}^{-1}\Bigg\{k!K_{n-m+k}(\lambda)\\ &+ \sum_{[\boldsymbol{m_{k-1}}]=k}\frac{k!}{f(\boldsymbol{m_{k-1}})}K^{(m_1+\cdots+m_{k-1})}_{n-m}(\lambda)\prod_{j=1}^{k-1}\Lambda_j^{m_j}\\ &+ \sum_{l=1}^{k-1}\binom{k}{l}\sum_{[\boldsymbol{m_{k-l}}]=k-l}\frac{(k-l)!}{f(\boldsymbol{m_{k-l}})}K^{(m_1+\cdots+m_{k-l})}_{n-m+l}(\lambda)\\ &\prod_{j=1}^{k-l}\Lambda_j^{m_j}l!\Bigg\}\end{aligned} \tag{29}$$

where $\sum_{[\boldsymbol{m_i}]=j}$ indicates the summation for $i$-tuples of non-negative integers $\boldsymbol{m_i} = (m_1,\cdots,m_i)$ that satisfy

$$1\cdot m_1 + 2\cdot m_2 + \cdots + i\cdot m_i = j \tag{30}$$

and $f(\boldsymbol{m_i}) = m_1!1!^{m_1}m_2!2!^{m_2}\cdots m_i!i!^{m_i}$.

*Proof:* The left-hand side of the identity $\frac{\partial^k}{\partial\tau^k}\Phi(\gamma(\tau),\tau) \equiv 0$ is expressed as

$$\begin{aligned}&\sum_{i=0}^{\infty}\sum_{l=0}^{k-1}\left\{\binom{k}{l}\left(\frac{d^{k-l}}{d\tau^{k-l}}K_{n-m+i}(\gamma(\tau))\right)\frac{i!}{(i-l)!}\delta_i^{\max(l,i)}\tau^{i-l}\right\}\\ &+\sum_{i=0}^{\infty}\left\{\binom{k}{k}K_{n-m+i}(\gamma(\tau))\frac{i!}{(i-k)!}\delta_i^{\max(l,i)}\tau^{i-k}\right\}\end{aligned} \tag{31}$$

where

$$\delta_i^{\max(l,i)} = \begin{cases} 1 & (i \geq l) \\ 0 & (i < l)\end{cases} \tag{32}$$

By Faà di Bruno's formula for the generalized chain rule for higher derivatives [7], expression (31) yields

$$\sum_{i=0}^{\infty}\sum_{l=0}^{k-1}\left[\left(\begin{array}{c}k\\ l\end{array}\right)\left\{\sum_{[\boldsymbol{m_{k-l}}]=k-l}\frac{(k-l)!}{f(\boldsymbol{m}_{k-l})}K_{n-m+i}^{(m_1+\cdots+m_{k-l})}(\gamma(\tau))\prod_{j=1}^{k-l}\left(\frac{d^j}{d\tau^j}\gamma(\tau)\right)^{m_j}\right\}\frac{i!}{(i-l)!}\delta_i^{\max(l,i)}\tau^{i-l}\right] + \left(\begin{array}{c}k\\ k\end{array}\right)F_k(\gamma(\tau))\frac{k!}{(k-k)!} \quad (33)$$

By setting $\tau=0$, expression (33) yields

$$\sum_{l=1}^{k-1}\left(\begin{array}{c}k\\ l\end{array}\right)\sum_{[\boldsymbol{m}_{k-l}]=k-l}\frac{(k-l)!}{f(\boldsymbol{m}_{k-l})}K_{n-m+l}^{(m_1+\cdots+m_{k-l})}(\lambda)\prod_{j=1}^{k-l}\Lambda_j^{m_j}l! + \left(\begin{array}{c}k\\ 0\end{array}\right)\sum_{[\boldsymbol{m}_k]=k}\frac{k!}{f(\boldsymbol{m}_k)}K_{n-m}^{(m_1+\cdots+m_k)}(\lambda)\prod_{j=1}^{k}\Lambda_j^{m_j} + k!K_{n-m+k}(\lambda) \quad (34)$$

Note that in expression (34), $\Lambda_k$ emerges only in the second term, and the only combination for $m_k\neq 0$ that satisfies $[\boldsymbol{m}_k]=k$ is $\boldsymbol{m}_k=(0,\cdots,0,1)$. By separating the summation into $\boldsymbol{m}_k=(0,\cdots,0,1)$ and $(m_1,\cdots,m_{k-1},0)$, we obtain

$$\sum_{l=1}^{k-1}\left(\begin{array}{c}k\\ l\end{array}\right)\sum_{[\boldsymbol{m}_{k-l}]=k-l}\frac{(k-l)!}{f(\boldsymbol{m}_{k-l})}K_{n-m+l}^{(m_1+\cdots+m_{k-l})}(\lambda)\prod_{j=1}^{k-l}\Lambda_j^{m_j}l! + \sum_{[\boldsymbol{m}_{k-1}]=k}\frac{k!}{f(\boldsymbol{m}_{k-1})}K_{n-m}^{(m_1+\cdots+m_{k-1})}(\lambda)\prod_{j=1}^{k-1}\Lambda_j^{m_j} + K_{n-m}^{(1)}(\lambda)\Lambda_k + k!K_{n-m+k}(\lambda)=0 \quad (35)$$

This completes the proof. ∎

Lemma 2 ensures that calculating the exact expressions of the equality $\frac{\partial^i}{\partial\tau^i}\Phi(\gamma(\tau),\tau)\Big|_{\tau=0}=0$ for $i=1,\cdots,k$ yields an explicit expression for coefficient $\Lambda_k$.

From Lemma 1 and 2, we conclude that each coefficient $\Lambda_k$ of the Taylor expansion is expressed as a linear combination of the monomials of $a_1,\cdots,a_n$ and $b_0,b_1,\cdots,b_m$ limited by a number that is related to the order $k$. Without loss of generality and for simplicity of expression, we assume that $b_0=1$.

*Theorem 1:* The coefficient $\Lambda_k$ in the Taylor expansion of discretization zeros (26) is expressed as

$$\Lambda_k=\sum_{[\boldsymbol{\nu}]+[\boldsymbol{\mu}]=k}C_k(\boldsymbol{\nu},\boldsymbol{\mu})a_1^{\nu_1}\cdots a_n^{\nu_n}\cdot b_1^{\mu_1}\cdots b_m^{\mu_m} \quad (36)$$

where $C_k(\boldsymbol{\nu},\boldsymbol{\mu})$ is a real constant, and $\sum_{[\boldsymbol{\nu}]+[\boldsymbol{\mu}]=k}$ indicates the summation for the $n$-tuple $\boldsymbol{\nu}=(\nu_1,\cdots,\nu_n)$ and $m$-tuple $\boldsymbol{\mu}=(\mu_1,\cdots,\mu_m)$ of nonnegative integers that satisfy

$$(1\cdot\nu_1+\cdots+n\cdot\nu_n)+(1\cdot\mu_1+\cdots+m\cdot\mu_m)=k \quad (37)$$

*Proof:* By Lemma 1, $K_{n-m+l}^{(\mu)}(\lambda)$ is represented for any $\mu$ as a linear combination of

$$a_1^{\nu_1}\cdots a_n^{\nu_n}b_j \quad (38)$$

where $n+1$-tuple $(\nu_1,\cdots,\nu_n,j)$ satisfies $1\cdot\nu_1+\cdots+n\cdot\nu_n+j=l$. Since $\Lambda_1=-K_{n-m+1}(\lambda)/K_{n-m}^{(1)}(\lambda)$, equation (36) is satisfied for $k=1$. We assume that equation (36) is satisfied for $\Lambda_j$, where $j=1,\cdots,k-1$. Then, $\prod_{j=1}^{\kappa}\Lambda_j^{m_j}$ is expressed as a linear combination of $a_1^{\nu_1}\cdots a_n^{\nu_n}\cdot b_1^{\mu_1}\cdots b_m^{\mu_m}$, where $[\boldsymbol{\nu}]+[\boldsymbol{\mu}]=1\cdot m_1+\cdots+\kappa\cdot m_\kappa=[\boldsymbol{m}_\kappa]$. Noting that the summations in equation (35) are limited to the combinations $[\boldsymbol{m}_{k-l}]=k-l$ and $[\boldsymbol{m}_{k-1}]=k$, we obtain equation (36) for $\Lambda_k$. By mathematical induction, we complete the proof. ∎

*Example 2:* Next, consider the case where $(n,m)=(3,1)$, as given in Example 1. There is one discretization zero $\gamma_2(\tau)$ that tends to the zero of $B_2(z)=z+1$, namely, $\lambda=-1$. By Theorem 1, we estimate the possible combinations of coefficients as $a_1$ and $b_1$ for $\Lambda_1$; $a_1^2$, $a_1b_1$ $b_1^2$, $a_2$ and $b_2$ for $\Lambda_2$; $a_1^3$, $a_1^2b_1$, $a_1a_2$, $a_2b_1$, $a_1b_1^2$, and $a_3$ for $\Lambda_3$. On the other hand, by applying the calculation results of equations (22) and (23), for example, to equations (27), (28), and (29), we obtain Taylor expansion (12), which is consistent with the above estimation.

### C. Taylor expansion for a single intrinsic zero

When $m=1$ or there is only one intrinsic zero, the intrinsic zero $\gamma_1(\tau)$ ($\to 1$ as $\tau\to 0$) is single and real for a sufficiently small sample time $\tau$. This fact, applied based on the implicit function theorem in Section II-B, implies that the single intrinsic zero $\gamma_1(\tau)$ can also be expressed by the Taylor expansion (26) with $\lambda=1$. A reasoning similar to the one in Section II-B leads to the following Theorem on the intrinsic zero, which corresponds to Lemma 2 and Theorem 1 on discretization zeros.

*Theorem 2:* When $m=1$, the coefficient $\Lambda_k$ for the Taylor expansion (26) of the single intrinsic zero $\gamma_1(\tau)$ is calculated using formula (29) with $\lambda=1$. The coefficient $\Lambda_k$ is expressed as formula (36) where the summation is restricted to the combination that satisfies equation (37).

*Example 3:* Applying the calculation formula (29) to the case $(n,m)=(3,1)$, we obtain the Taylor expansion as (13), which is consistent with the above estimation. It should be noted here that $K_6(z)$ and $K_7(z)$ are additionally prepared for the Taylor expansion of $\gamma_1(\tau)$ until $k=5$. The fact that the fifth term differs from that of the exponential function $\exp(-b_1\tau)$ is consistent with the known result on the intrinsic zero [6].

## III. Applications

The Taylor expansion (26) truncated at $k=M$ of any discretization zero or the single intrinsic zero is symbolically calculated using formula (29) after calculation of the symbolic expressions of $K_1(z),\cdots,K_{n-m+M}(z)$ by Procedure 1 presented at the beginning of Section II-A. Theorem 1 or 2 ensures that the effort needed for the symbolic calculation is less than that directly expected from the calculation formula. This result originated from Lemma 1, which ensures that the number of the possible terms[3] in the truncated expansion of numerator $N(z,\tau)$ or $\{K_1(z),\cdots,K_{n-m+M}(z)\}$ is limited to a number considerably smaller than the number that is generally estimated by formulae (4) and (6) for Procedure

[3]The number of terms in $K_k(z)$ is defined to be the number of terms that are summed in equation (16) substituted into equation (15) [8].

1. Since the number of combinations of nonnegative integers $(\nu_1, \cdots, \nu_n)$ in the summation of equation (16) based on the original definition is $\sum_{l=0}^{k} \frac{(l+n-1)!}{l!(n-1)!}$, in the worst case, the number of the terms in $\{K_1(z), \cdots, K_{n-m+M}(z)\}$ is estimated as

$$n(m+1) \sum_{k=1}^{n-m+M} \sum_{l=0}^{k} \frac{(l+n-1)!}{l!(n-1)!} \tag{39}$$

However, Lemma 1 guarantees that a considerably lesser number of terms than the worst-case number exist in the expression.

*Example 4:* Next, we consider the example given as Example 1, i.e., $(n, m) = (3, 1)$. When we truncate the expansion of the numerator $N(z, \tau)$ at $k = n - m + M$ where $M = 3$, the worst-case scenario with respect to the number of terms is estimated by equation (39) to be 750. However, as shown in Example 2, the number of possible combinations for equation (17), where $j = 0$ and 1, is 0 for $k = 0$ or 1, 1 for $k = 2$, 2 for $k = 3$, 3 for $k = 4$, and 5 for $k = 5$. The number of terms in the truncated numerator $N(z, \tau) = \beta_0(\tau)z^2 + \beta_1(\tau)z + \beta_2(\tau)$ is limited to at most $(1 + 2 + 3 + 5) \times 3 = 33$. The exact calculation by Procedure 1 results in (19), (20), and (21) where there exist $11 + 9 + 11 = 31$ terms, which is consistent with the above explanation.

*Example 5:* We consider the general transfer function (3), where $(n, m) = (8, 4)$ with $b_0 = 1$, and calculate the symbolic expressions of Taylor expansion (26) truncated at $k = M = 2$ of triple discretization zeros. For this purpose, we apply formula (29) in Lemma 2 to the symbolic calculation of $\Lambda_1$ and $\Lambda_2$ after preparing symbolic expressions of $K_k(z)$, where $k \leq 6$ $(= n - m + 2)$, by Procedure 1 with $k_{\max} = 6$. The worst-case number of terms in these polynomials $K_k(z)$ estimated by (39) explodes to 200,200. Fortunately, Lemma 1 guarantees that the number of terms remains at most 56 $(= 8 \times \{1 + (1 + 1) + (1 + 1 + 2)\})$ because combinations of nonnegative integers $(\nu_1, \cdots, \nu_8)$ satisfying (17) are $(0, \cdots, 0)$ for $k - (n - m) - j = 0$, $(1, 0, \cdots, 0)$ for $k - (n - m) - j = 1$, and $(2, 0, \cdots, 0)$ or $(0, 1, 0, \cdots, 0)$ for $k - (n - m) - j = 2$. However, since the computation of the determinant (6) at step 3 for Procedure 1 with $(A, B)$ prepared at step 2 with $k_{\max} = 6$ yields partial terms in unneeded $K_i(z)$ $(i = 7, 8, \cdots)$, the total number of generated terms runs up to 26,227, which should be reduced by some measure to prevent the useless terms from being generated. In fact, one of such measures are presented by Theorem 1 that implies that the objective expressions of discretization zeros is written as

$$\begin{aligned}&\lambda + \{C_1(1)a_1 + C_1(2)b_1\}\,\tau \\ &+ \left\{C_2(1)a_1^2 + C_2(2)a_1b_1 + C_2(3)b_1^2 + C_2(4)a_2 \right.\\ &\left.+ C_2(5)b_2\right\}\tau^2/2\end{aligned} \tag{40}$$

where $\lambda$ is one of the zeros of $B_4(z) = z^3 + 11z^2 + 11z + 1$ and $C_i(j)$ are real constants. Hence, $a_i$ or $b_i$, where $i \geq 3$, are not needed for our purpose. By replacing unneeded symbols $\{a_i, b_i | i \geq 3\}$ with 0 in the matrices $(A_c, B_c, C)$ in advance of step 2 for Procedure 1, we can reduce the effort required at step 2 and 3. The number of terms generated by the computation after eliminating these unneeded symbols was found to be 424. The expressions to be applied to formula (29) to yield $\Lambda_1$ and $\Lambda_2$ are chosen from 424 terms of the above-mentioned calculation result as follows:

$$\begin{aligned}K_4(z) =& \frac{1}{4!}\left\{z^7 + 7\,z^6 - 27\,z^5 + 19\,z^4 \right.\\ &\left. + 19\,z^3 - 27\,z^2 + 7\,z + 1\right\}\\ =& \frac{1}{4!}(z+1)(z^2+10z+1)(z-1)^4\end{aligned} \tag{41}$$

$$\begin{aligned}K_5(z) =& \frac{1}{5!}\left\{(-a_1 + b_1)z^7 + (-18\,a_1 + 23\,b_1)z^6 \right.\\ &+ (49\,a_1 - 9\,b_1)z^5 - 95\,z^4 b_1 + (-95\,a_1 + 95\,b_1)z^3\\ &\left. + (86\,a_1 + 9\,b_1)z^2 + (-17\,a_1 - 23\,b_1)z - 4\,a_1 - b_1\right\}\end{aligned} \tag{42}$$

$$\begin{aligned}K_6(z) =& \frac{1}{6!}\left\{({a_1}^2 - a_1b_1 - a_2 + b_2)z^7 \right.\\ &+ (34\,{a_1}^2 - 49\,a_1b_1 - 25\,a_2 + 55\,b_2)z^6\\ &+ (-75\,{a_1}^2 - 45\,a_1b_1 + 81\,a_2 + 189\,b_2)z^5\\ &+ (-50\,{a_1}^2 + 335\,a_1b_1 - 55\,a_2 - 245\,b_2)z^4\\ &+ (235\,{a_1}^2 - 235\,a_1b_1 - 55\,a_2 - 245\,b_2)z^3\\ &+ (-186\,{a_1}^2 - 99\,a_1b_1 + 81\,a_2 + 189\,b_2)z^2\\ &+ (31\,{a_1}^2 + 89\,a_1b_1 - 25\,a_2 + 55\,b_2)z\\ &\left. + 10\,{a_1}^2 + 5\,a_1b_1 - a_2 + b_2\right\}\end{aligned} \tag{43}$$

Taylor expansions of the discretization zeros are yielded as follows:

$$\gamma_1(\tau) = -1 + \frac{1}{5}(a_1 - b_1)\tau - \frac{1}{25}(a_1^2 - 2a_1b_1 + b_1^2)\tau^2/2 + \cdots \tag{44}$$

$$\begin{aligned}\gamma_2(\tau) =& -5 - 2\sqrt{6} + \frac{49 + 20\sqrt{6}}{25 + 10\sqrt{6}}(a_1 - b_1)\tau\\ &+ \frac{1}{100(5 + 2\sqrt{6})^3(3 + \sqrt{6})}\left\{\right.\\ &(-95050\sqrt{6} - 232824)a_1^2 + (59285\sqrt{6} + 145218)a_1b_1\\ &+ (35765\,\sqrt{6} + 87606)b_1^2 + (130815\,\sqrt{6} + 320430)a_2\\ &\left.(-130815\,\sqrt{6} - 320430)b_2\right\}\tau^2/2 + \cdots\end{aligned} \tag{45}$$

$$\begin{aligned}\gamma_3(\tau) =& -5 + 2\sqrt{6} + \frac{-49 + 20\sqrt{6}}{-25 + 10\sqrt{6}}(a_1 - b_1)\tau\\ &+ \frac{1}{100(-5 + 2\sqrt{6})^3(-3 + \sqrt{6})}\left\{\right.\\ &(95050\sqrt{6} - 232824)a_1^2 + (-59285\sqrt{6} + 145218)a_1b_1\\ &+ (-35765\,\sqrt{6} + 87606)b_1^2\\ &+ (-130815\,\sqrt{6} + 320430)a_2\\ &\left.(130815\,\sqrt{6} - 320430)b_2\right\}\tau^2/2 + \cdots\end{aligned} \tag{46}$$

Since the total number of terms in expressions (41), (42) and (43) is 55, the above-mentioned computation with elimination of unneeded symbols has still yielded $424 - 55 = 369$ unneeded terms, which consist only of $\{a_1, a_2, b_0, b_1, b_2\}$ but should be parts of unneeded $K_i(z)$ $(i \geq 7)$. Note, however, that the number of unneeded terms avoided by the elimination of unneeded symbols in advance is $26{,}227 - 424 = 25{,}803$, which is much larger than $424 - 55 = 369$. Moreover, we note that preventing such 25,803 terms from being generated saves much computing time. Table I shows a comparison of computing time[4] used for determinant computation at Step 3 for Procedure 1 with the following different realizations.

Case 1: Controllable canonical form $(\bar{A}_c, \bar{B}_c, \bar{C})$ which is one of the most sparse matrix realization.

Case 2: Similar-transformed controllable canonical form $(A_c, B_c, C) = (T_2^{-1}\bar{A}_c T_2, T_2^{-1}\bar{B}_c, \bar{C}T_2)$ by

$$T_2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} \quad (47)$$

Every element in $(A_c, B_c, C)$ is non-zero.

| | case 1 | case 2 |
|---|---|---|
| without elimination (26,227 terms generated) | 8.441 | $5.3718 \times 10^5$ |
| with elimination (424 terms generated) | 0.017 | 1.635 |

TABLE I
COMPUTING TIME USED FOR DETERMINANT CALCULATION

Even for the sparse matrix realization (case 1), the computation time is reduced to around $1/500$. The above results demonstrate the advantage of using Theorem 1 for computation of Taylor expansion when the order $n$ or $m$ of the system is large.

## IV. CONCLUSION

The paper introduces key properties useful for the computation of Taylor series expansion of any discretization zero or single intrinsic zero with regularity of the suffixes of the coefficient. The regularity is shown to reduce the calculation effort for higher-order systems. The regularity also indicates that the discretization zeros, single intrinsic zero, or every coefficient in the numerator of the transfer function of the sampled-data systems are dominated by the coefficients for higher-order term $s^k$ in the numerator and denominator of the continuous-time counterpart.

[4]CPU time [sec] by symbolic computing software Maple on Mac Pro 3.5GHz 6-Core Intel Xeon E5 with 16GB memory

## APPENDIX A
## PROOF OF LEMMA 1

First, we prepare some equations to be used for proof of Lemma 1. By equations (2) and (3), we have

$$(-1)^1 a_1 = p_1 + \cdots + p_n \quad (48)$$

$$(-1)^2 a_2 = p_1 p_2 + \cdots + p_{n-1} p_n \quad (49)$$

$$\vdots$$

$$(-1)^j a_j = \sum_{1 \le i_1 < \cdots < i_j \le n} p_{i_1} p_{i_2} \cdots p_{i_j} \quad (50)$$

$$\vdots$$

$$(-1)^n a_n = p_1 p_2 \cdots p_n \quad (51)$$

The sum in equation (50) is for $j$-tuples of integers $(i_1, \cdots, i_j)$ that satisfy $1 \le i_1 < \cdots < i_j \le n$ Note that, from the viewpoint of the multivariate polynomial of $p_1, \cdots, p_n$, equations (48) through (51) are referred to as elementary symmetric polynomials [8]. In the following, we denote the $j$-th elementary symmetric polynomial defined by (50) of the $n$ variables $\{p_1, \cdots, p_n\}$ as $\sigma_{n,j}(\{p_1, \cdots, p_n\})$. Note here that the total degree of polynomial $\sigma_{n,j}(\{p_1, \cdots, p_n\}) = (-1)^j a_j$ denoted by $\deg(\sigma_{n,j}) = \deg(a_j)$ is equal to the suffix $j$.

Here, using the expression

$$\frac{G(s)}{s} = \frac{r_0}{s} + \frac{r_1}{s - p_1} + \cdots + \frac{r_n}{s - p_n} \quad (52)$$

we prepare another expression of the transfer function [1]

$$H(z) = (1 - z^{-1})\mathcal{Z}\left[\frac{G(s)}{s}\right] \quad (53)$$

$$= \frac{z-1}{z}\left\{\frac{r_0 z}{z - \exp(p_0\tau)} + \cdots + \frac{r_n z}{z - \exp(p_n\tau)}\right\} \quad (54)$$

where $\mathcal{Z}[\cdot]$ indicates z-transform after inverse Laplace transform and sampling; we define $p_0 = 0$. From equations (2) and (52), we obtain the equations

$$(-1)^{j-(n-m)} b_{j-(n-m)} = \sum_{l=0}^{n} r_l \sigma_{n,j}(\{p_0, p_1, \cdots, p_n\} \backslash \{p_l\}) \quad (55)$$

where $j = n - m, \cdots, n$, and the set $\{p_0, p_1, \cdots, p_n\} \backslash \{p_l\}$ indicates the set of $n+1$ variables $\{p_0, p_1, \cdots, p_n\}$ excluding $\{p_l\}$, which is a set of $n$ variables. For the sake of convenience, in the following discussions we define $b_{j-(n-m)} = 0$ for $j = 1, \cdots, (n-m) - 1$. From equations (7) and (54), we have

$$(-1)^j \beta_j = \sum_{l=0}^{n} r_l \sigma_{n,j}(\{\exp(p_0\tau), \cdots, \exp(p_n\tau)\} \backslash \{\exp(p_l\tau)\}) \quad (56)$$

Noting that

$$\sigma_{n,j}(\{\exp(p_1\tau), \cdots, \exp(p_n\tau)\})$$

$$= \sum_{1 \le i_1 < \cdots < i_j \le n} \exp((p_{i_1} + \cdots + p_{i_j})\tau) \quad (57)$$

$$= \sum_{k=0}^{\infty} \frac{\tau^k}{k!} \sum_{1 \le i_1 < \cdots < i_j \le n} (p_{i_1} + \cdots + p_{i_j})^k \quad (58)$$

equation (56) yields

$$(-1)^j\beta_j = \sum_{k=0}^{\infty}\frac{\tau^k}{k!}\sum_{l=0}^{n} r_l S_{j,k}(\{p_0, p_1, \cdots, p_n\}\backslash\{p_l\}) \quad (59)$$

where

$$S_{j,k}(\{p_0, p_1, \cdots, p_n\}\backslash\{p_l\}) = \sum_{0\le i_1<\cdots<i_j\le n}^{(\neq l)} (p_{i_1}+\cdots+p_{i_j})^k \quad (60)$$

The sum $\sum_{0\le i_1<\cdots<i_j\le n}^{(\neq l)}$ is the sum of $j$-tuples of integers $(i_1, \cdots, i_j)$ that satisfy $i_1 \neq l, \cdots, i_n \neq l$ and $0 \le i_1 < \cdots < i_j \le n$. Note that the multivariate polynomial $S_{j,k}$ previously defined is a symmetric polynomial, which is identical for every possible permutation of the variables [8]. In order to express equation (59) in terms of equations (48) through (55), we note the following:

*Proposition 1:* [8] Every symmetric polynomial of $n$ variables can be written uniquely as a polynomial of the elementary symmetric polynomials of the $n$ variables.

By this proposition, we have

$$\begin{aligned} &S_{j,k}(\{p_0, p_1, \cdots, p_n\}\backslash\{p_l\}) \\ &= \sum_{(\mu_1,\cdots,\mu_n)} c_{j,k}(\mu_1, \cdots, \mu_n)\sigma_{n,1,l}^{\mu_1}\cdots\sigma_{n,n,l}^{\mu_n} \end{aligned} \quad (61)$$

where the sum is for $n$-tuples of nonnegative integers $(\mu_1, \cdots, \mu_n)$, $c_{j,l}(\mu_1, \cdots, \mu_n)$ are constants, and $\sigma_{n,1,l}, \ldots, \sigma_{n,n,l}$ indicate the elementary symmetric polynomials of $n$ variables $\{p_0, p_1, \cdots, p_n\}\backslash\{p_l\}$. Next, we note that the expressions of the elementary symmetric polynomial, i.e., (48), $\cdots$, (51) yield the following identities:

$$\sigma_{n+1,1} = \sigma_{n,1,l} + p_l \quad (62)$$

$$\sigma_{n+1,2} = \sigma_{n,2,l} + p_l\sigma_{n,1,l} \quad (63)$$

$$\vdots$$

$$\sigma_{n+1,n} = \sigma_{n,n,l} + p_l\sigma_{n,n-1,l} \quad (64)$$

$$\sigma_{n+1,n+1} = p_l\sigma_{n,n,l} \quad (65)$$

Using these identities, we prepare a key lemma:

*Lemma 3:* Any monomial of elementary symmetric polynomials $\sigma_{n,1,l}^{\mu_1}\cdots\sigma_{n,n,l}^{\mu_n}$ of $n$ variables $\{p_0, p_1, \cdots, p_n\}\backslash\{p_l\}$ can be written for any $l \in \{0, \cdots, n\}$ by using the elementary symmetric polynomials $\{\sigma_{n+1,1}, \cdots, \sigma_{n+1,n+1}\}$ of the $n+1$ variables, $\{p_0, p_1, \cdots, p_n\}$ excluding no elements as

$$\begin{aligned} &\sigma_{n,1,l}^{\mu_1}\cdots\sigma_{n,n,l}^{\mu_n} \\ &= \sum_{\eta=1}^{n}\sigma_{n,\eta,l}\sum_{(\nu_1,\cdots,\nu_{n+1})} c_\eta(\nu_1, \cdots, \nu_{n+1})\sigma_{n+1,1}^{\nu_1}\cdots\sigma_{n+1,n+1}^{\nu_{n+1}} \end{aligned} \quad (66)$$

where $c_\eta(\nu_1, \cdots, \nu_{n+1})$ are constants.

*Proof:* We prepare the following identities:

$$\sigma_{n,k,l} = \sum_{i=0}^{k}(-1)^i\sigma_{n+1,k-i}p_l^i \quad (k = 1, \cdots, n) \quad (67)$$

$$p_l^{n+1} = (-1)^n\sum_{i=0}^{n}(-1)^i\sigma_{n+1,n+1-i}p_l^i \quad (68)$$

$$\begin{aligned} p_l^k =& (-1)^k\left\{\sigma_{n,k,l} - \sum_{i=0}^{k-1}(-1)^i\sigma_{n+1,k-i}p_l^i\right\} \\ &(k = 1, \cdots, n) \end{aligned} \quad (69)$$

where we define $\sigma_{n+1,0} = 1$ for expressive simplicity. Identity (67) is derived by recursive substitution regarding identities (62) through (64) such as the substitution of equation (62) into equation (63) to eliminate the term $\sigma_{n,1,l}$. Identity (68) is derived by substituting identity (67) and $\sigma_{n+1,0} = 1$ into identity (65). Identity (69) is equivalent to identity (67) rewritten using $\sigma_{n+1,0} = 1$. Next, we apply identities (67), (68), and (69) to the left-hand side of equation (66), which is to be proven. First, we use identity (67) to eliminate $\sigma_{n,1,l}, \cdots, \sigma_{n,n,l}$ and then substitute identity (68) repeatedly in order to decrease the order of $p_l$ until the maximum order becomes less than $n+1$:

$$\begin{aligned} &\sigma_{n,1,l}^{\mu_1}\cdots\sigma_{n,n,l}^{\mu_n} \\ &= \sum_{k=1}^{n}\left\{\sum_{(\nu_1,\cdots,\nu_{n+1})} C_k(\nu_1, \cdots, \nu_{n+1})\sigma_{n+1,1}^{\nu_1}\cdots\sigma_{n+1,n+1}^{\nu_{n+1}}\right\}p_l^k \end{aligned} \quad (70)$$

where $\sum_{(\nu_1,\cdots,\nu_{n+1})}$ indicates the summation for $n+1$-tuples of nonnegative integers $(\nu_1, \cdots, \nu_{n+1})$ and $C_k(\nu_1, \cdots, \nu_{n+1})$ are integer constants. Substituting identity (69) with $k = n$ into equation (70), we obtain

$$\begin{aligned} &\sigma_{n,1,l}^{\mu_1}\cdots\sigma_{n,n,l}^{\mu_n} \\ &= \left\{\sum_{(\nu_1,\cdots,\nu_{n+1})} C_n(\nu_1, \cdots, \nu_{n+1})\sigma_{n+1,1}^{\nu_1}\cdots\sigma_{n+1,n+1}^{\nu_{n+1}}\right\} \\ &\quad (-1)\sigma_{n,n,l} + \sum_{k=1}^{n-1} \\ &\quad \left\{\sum_{(\nu_1,\cdots,\nu_{n+1})} \bar{C}_k(\nu_1, \cdots, \nu_{n+1})\sigma_{n+1,1}^{\nu_1}\cdots\sigma_{n+1,n+1}^{\nu_{n+1}}\right\}p_l^k \end{aligned} \quad (71)$$

where $\bar{C}_k(\nu_1, \cdots, \nu_{n+1})$ are the integer constants calculated from constants $C_k(\nu_1, \cdots, \nu_{n+1})$ with $k = n, \cdots, 1$. Repeating the substitution of identity (69) with $k = n-1, \cdots, 1$ sequentially, we eliminate $p_l^{n-1}, \cdots, p_l^1$ and establish Lemma 3. ■

*Remark 1:* Note that Lemma 3 holds independently of the above-mentioned assumption, i.e., $p_0 = 0$, and holds identically for any values of $(p_0, \cdots, p_n)$. Moreover, the summation for $n+1$-tuples $(\nu_1, \cdots, \nu_{n+1})$ in equation (66) is independent of suffix $l$ of the excluded variable $p_l$. Hence, we have

$$\sum_{l=0}^{n} r_l \sigma_{n,1,l}^{\mu_1} \cdots \sigma_{n,n,l}^{\mu_n} = \sum_{\eta=1}^{n} \sum_{(\nu_1,\cdots,\nu_{n+1})} c_\eta(\nu_1,\cdots,\nu_{n+1}) \sigma_{n+1,1}^{\nu_1} \cdots \sigma_{n+1,n+1}^{\nu_{n+1}} \sum_{l=0}^{n} r_l \sigma_{n,\eta,l} \quad (72)$$

where we changed the order of summation, making that over $l$ innermost.

Now, we prove Lemma 1. By equations (61) and (72), equation (59) yields

$$(-1)^j \beta_j = \sum_{k=0}^{\infty} \frac{\tau^k}{k!} \sum c_{j,k} \sum_{\eta=1}^{n} \sum_{(\nu_1,\cdots,\nu_{n+1})} c_\eta \sigma_{n+1,1}^{\nu_1} \cdots \sigma_{n+1,n+1}^{\nu_{n+1}} \sum_{l=0}^{n} r_l \sigma_{n,\eta,l} \quad (73)$$

Since we defined $p_0 = 0$, we have $\sigma_{n+1,i}(\{p_0, p_1, \cdots, p_n\}) = \sigma_{n,i}(\{p_1, \cdots, p_n\})$ for $i = 1, \cdots, n$ and $\sigma_{n+1,n+1} = 0$. By substituting equations (48) through (55) into equation (73), we obtain

$$(-1)^j \beta_j = \sum_{k=0}^{\infty} \frac{\tau^k}{k!} \sum c_{j,k} \sum_{\eta=1}^{n} \sum_{(\nu_1,\cdots,\nu_{n+1})} \bar{c}_\eta \{(-1)^1 a_1\}^{\nu_1} \cdots \{(-1)^n a_n\}^{\nu_n} (-1)^{\eta-(n-m)} b_{\eta-(n-m)} \quad (74)$$

where $\bar{c}_\eta(k, \nu_1, \cdots, \nu_n) = c_\eta(k, \nu_1, \cdots, \nu_n, 0)$. Expression (74) for $\beta_j$ is equivalent to that obtained by comparing equations (14), (15), and (16) with the definition given in equation (7).

By comparing the terms of $\tau^k$ in the right-hand sides of equations (74) and (59), we obtain

$$\sum_{l=0}^{n} r_l S_{j,k}(\{p_0, p_1, \cdots, p_n\} \backslash \{p_l\}) = 0 \quad (75)$$

for $k = 0, \cdots, n-m-1$ because $b_{\eta-(n-m)} = 0$ for $\eta = 1, \cdots, (n-m)-1$ and $\deg\left(\sum_{l=0}^{n} r_l \sigma_{n,\eta,l}\right) = \eta$, which implies that $\bar{c}_\eta = 0$ for $\eta > \deg(S_{j,k}) = k$, where $\deg$ indicates the total degree in terms of the multivariate polynomial of $\{p_0, p_1, \cdots, p_n\}$. Moreover, we obtain

$$\deg\left(\sum_{l=0}^{n} r_l S_{j,k}(\{p_0, p_1, \cdots, p_n\} \backslash \{p_l\})\right) = \deg(b_{\eta-(n-m)}) + \deg(a_1) \cdot \nu_1 + \cdots + \deg(a_n) \cdot \nu_n \quad (76)$$

for $k \geq n + m$. Since we have $\deg(b_{j-(n-m)}) = j$ and $\deg(a_j) = j$ from equations (55) and (50), we complete the proof of Lemma 1.

## References


[1] G. F. Franklin, J. D. Powell, and M. L. Workman, *Digital Control of Dynamic Systems*. Addison–Wesley, 1998.

[2] K. J. Åström, P. Hagander, and J. Sternby, "Zeros of sampled systems," *Automatica*, vol. 20, no. 1, pp. 31–38, 1984.

[3] S. R. Weller, W. Moran, B. Ninness, and A. D. Pollington, "Sampling zeros and the Euler–Frobenius polynomials," *IEEE Transaction on Automatic Control*, vol. 46, no. 2, pp. 340–343, 2001.

[4] T. Hagiwara and M. Araki, "Stability of the limiting zeros of sampled-data systems with zero and first-order holds," *Int. J. Control*, vol. 58, no. 6, pp. 1325–1346, 1993.

[5] T. Hagiwara, "Analytic study on the intrinsic zeros of sampled-data systems," *IEEE Transactions on Automatic Control*, vol. 41, no. 2, pp. 261–263, 1996.

[6] M. Blachuta, "On zeros of sampled systems," *IEEE Transactions on Automatic Control*, vol. AC-44, pp. 1229–1234, 1999.

[7] S. Roman, "The formula of Faà di Bruno," *American Mathematical Monthly*, vol. 87, pp. 805–809, 1980.

[8] D. A. Cox, J. Little, and D. O'Shea, *Ideals, Varieties, and Algorithms: An Introduction to Computational Algebraic Geometry and Commutative Algebra*. Springer, 2007.